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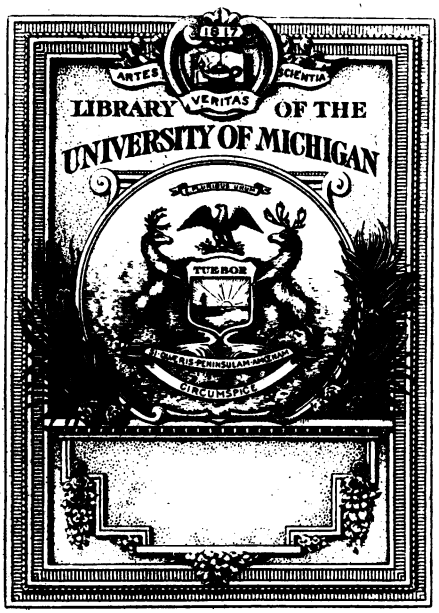

THE GIFT OF
Prof. Louis Karpinski

PLANE AND SOLID GEOMETRY

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EDITOR'S PREFACE

This text derives its origin and character from the view that geometry is a valuable study for high school pupils in the degree to which they understand and appreciate it at the time they are studying it. Appreciation of the worth of what is being done and insight into the meaning of its tasks go very far toward a real motivation of geometry. The system of thought it embodies is of course the chief claim of geometry to a place in the curriculum, but this system of thought is of no great value to one who neither appreciates nor understands its spirit.

Furthermore, this system of thought is a *derived* product. It arises from the habits formed from doing things geometrically and reasoning about what is done and why it is so done. It must accordingly wait upon many an act of intelligent judging and discriminating. The acts of systematic judging and discerning find a rich genetic background in measurement, in constructive exercises, in comparison of figures, and in geometrical experimentation by the student. This text so organizes the material as to give a concrete, experimental, and somewhat informal approach to the rather highly wrought scheme of demonstrative geometry. This informality pervades the first half of the first chapter, thereby laying a firm conceptual basis for the more systematic geometry. Throughout the text, however, subjects of special difficulty to high school students are approached experimentally. Mental possibilities are nowhere sacrificed to logical refinements. *Logical accuracy that is beyond the reach of the learner is held to be only apparent and largely specious.*

Besides the closer conformity to pedagogical standards of this inductive and informal approach to systematic geometry,

there is also an important *scientific* gain from it. It is commonly regarded as bad scientific practice to allow unlimited use of hypothetical constructions. With beginners in geometry it is always a question how far it is desirable to permit them to give proofs that involve constructions which they have not yet learned. For example, before the student has learned how to bisect an angle he may be allowed, openly or tacitly, to assume that a bisector can be drawn, while he is proving the theorem about the base angles of an isosceles triangle. The real question is how far shall this sort of practice be allowed. The prevailing view is that it should be practiced no longer than necessary. Too long continued, it blunts the learner's perception of the nature of real geometrical proof. By including in the preliminary work such constructions as are needed early in proofs, the need for hypothetical constructions is reduced to a minimum. The present organization thus scores a real gain in soundness of early geometrical thinking in the very act of reaching an important pedagogical end.

To the beginner no motives are so wholesomely appealing as the all-around usefulness of the study. "What benefit can be derived from the subject?" is the question most on his mind and lips. A good pedagogical organization must so marshal its ideas for presentation that the most attractive ones are in the foreground. This important principle has been kept prominent in the authors' minds in the organization of the subject as a whole, and also of its important parts. The exercise lists are exceptionally rich and varied in really practical problems, without at any point failing to include an ample supply of the standard geometric type as well. Many of the abstract exercises are followed at once by practical problems that give a life-setting to the geometric principles involved. An earnest attempt has been made to give a great variety to the types of practical exercise. This feature will assist in reaching a wide range of taste and aptitude among pupils, and also in keeping up the spirit of study through a many-sided motivation. This

cannot fail to appeal to those who hold the view that it is of maximum importance to make the learner's work seem worth while to him while he is yet in school.

The text lays no claim to logical precision beyond that which good students can appreciate. It recognizes that boys and girls can *prove* things long before thay can *demonstrate* them. That is to say, they can bring to bear enough relevant evidence to heighten materially the feeling of certainty of the truth of a theorem before they can intelligently master the technique of formal deductive proof. This treatment recognizes the learner's right to acquire the ability to demonstrate through the exercise of his ability to prove. To do otherwise is to endanger needlessly the very spirit of geometric reasoning in a vain show of mechanical steps and logical technique. In gauging this treatment, pedagogical rather than logical standards should be applied. *Logic has been intentionally sacrificed to insight whenever it was believed that the general geometric interests of the student would be thereby materially subserved.*

As *mental training* the value of solid geometry consists in the excellent culture it affords the space intuitions and the powers of space imagery. The close conformity of its subject-matter to the three-dimensional world in which the pupil lives, no less than the measure of skill he has acquired in the study of plane geometry in both inference-*making* and inference-*guiding*, gives to solid geometry, as here presented, a preëminent rank as space-training.

As *practical training* the value of solid geometry consists in its scheme of organization of the world of space and magnitude under a few standard forms, such as the plane, prism, pyramid, cylinder, cone, sphere, etc. *Mensuration* much more largely than *demonstration* motivates student activity in high school geometry, and mensuration of surfaces and volumes seems to the student worth learning. When a goodly proportion of the forms measured and calcu-

lated are taken from life-situations, as is done by the authors of this book, solid geometry acquires preëminent worth as *practical training.*

The present treatment of solid geometry will, it is believed, secure both these phases of training.

The INTRODUCTION to Solid Geometry, pp. 273-276, only illustrates the scrupulous care with which the authors, throughout the treatment, have striven to overcome a most persistent difficulty of students to understand and properly to interpret diagrams of solids. The authors fully realize the inability of many students to imagine clearly the spatial forms that the diagrams represent, and have shown great skill, by the aid of models, pictures, etc., in giving the necessary help, without over-helping.

Finally, this book has been prepared by actual teachers of long and successful high school experience. They know high school boys and girls. Visionary theories about what ought to be, but cannot be, have been ruthlessly set aside. They have kept an eye single to what is feasible, practicable, and remunerative in the class-room. Much of the manuscript was used for years in mimeograph form and the roughnesses have been smoothed off through class-room findings. The plan has also worked successfully in the hands of other teachers than the author's and in much less practicable form than that in which it is here given to the public. The editor feels that the text has an important rôle to play in the attempt now being made to bring school geometry into closer conformity with the needs, standards, and possibilities of those who are to study it.

Chicago, April, 1919. G. W. MYERS.

AUTHORS' PREFACE

The purpose of this book is to present the essentials of geometry together with some of their applications. There is no subject in the high school curriculum which can be made of more practical value to the average student. The work has been so arranged as to give suitable mathematical training as well as to appeal to the vital interests of the growing intellect.

The main aim of the authors in the preparation of this text has been to approach abstract reasoning by a method that is natural and comprehensible to the youthful mind, and to vitalize the subject-matter,— making it both interesting and useful through a wide range of practical applications.

The plans which have been adopted for the attainment of this end are outlined in the following paragraphs.

1. The method of presentation is psychological rather than that of pure deductive logic. While most of the subject is presented in the traditional form, the experimental or inductive method is employed with some of the fundamental propositions, especially in the early pages. In this way, the student is enabled to comprehend the exact meaning of these propositions and to apply them at once to the solution of exercises. Later, after he has acquired some skill in using the method of formal logic, all of these theorems are proved deductively.

2. Actual work in geometry is begun at once without a formidable array of definitions, axioms, and principles. In the Introduction, only an informal treatment of terms is given and only such topics are discussed as are necessary for the study of the early propositions and exercises. It should be noted that no attempt is made at this point to give a technical definition of any of the terms considered. Funda-

mental definitions and basic discussions are presented in the latter part of the first chapter. This plan avoids the confusion arising from the presentation of abstract material which the pupil does not comprehend at this stage, but which causes him little difficulty after he has had a partial survey of the subject.

Axioms and basic principles are introduced experimentally before they are expressed in a formal statement. In general, definitions and axioms are placed in the text at the point where they are first needed.

3. Especial attention has been given to the selection and arrangement of original exercises.

(*a*) They are introduced almost at the beginning and are very carefully graded. At first a few are proved in detail, in order that the student may understand what is desired in the solution of exercises and to give a model for writing out the proof.

(*b*) In the first chapter, a large proportion of the exercises are accompanied by figures. This enables the student to concentrate his attention upon the proof and to economize time in solving exercises. It also makes possible the solution of many exercises at sight during the recitation, a training which develops rapid thinking through concerted effort among the members of the class. As the work advances, the student is required with increasing frequency to construct the figures from the text.

(*c*) The large number and great variety of exercises are carefully distributed and range from some quite elementary to others of considerable difficulty. This enables the teacher to adapt the book to any group of students, whether in a classical or a technical high school. The abstract exercises give mental training and application of the basic theorems, while the practical exercises are used to correlate geometric facts with real life. For the most part, the problems pertaining to shop work are for use in technical schools. The

exercises marked with a star (*) are usually more difficult or of special application.

(*d*) Many of the exercises involve an application of arithmetic and algebra to geometry. Formulas are given in connection with the theorems and form the basis of many literal exercises and of much computation.

4. The proof of the most difficult as well as of the early theorems is given in full, but, as the work progresses, a gradual elimination of parts of proofs is made as the needs and capacity of the learner increase. Thus a middle course is adopted in the use of the suggestive method. In this way, the student thoroughly comprehends the subject-matter and at the same time is strengthened by doing work within his grasp.

5. The work has been planned so that practical applications may be made as early in the course as possible. To this end, the chapter on areas is placed before that on similarity because it furnishes a great variety of applications to real life.

The traditional division into parts is made, but these are named "chapters" because the word "book" has only historical value in this connection.

In addition to the five cardinal points outlined above, attention is called to the following features of the text.

To a great extent, terms and expressions are used which carry their own meaning. For example, "equal" is used instead of "congruent," at first because the word requires no explanation and adds no obstacle to the beginner. Some words like "corollary," "homologous," and "scholium," which are of doubtful value, are either used sparingly or omitted.

All basic material for meeting college requirements or for use in advanced mathematics has been given adequate treatment. A few topics, such as the steel square, the plane-table, screw-threads, and trigonometric ratios, are included. These are to be studied as occasion may demand.

The experiments may be performed on the laboratory or supervised study plan, with the pupils working under the guidance of the teacher. This makes possible a thorough understanding of what is expected and the attainment of most satisfactory results. For convenience, a combination ruler and protractor accompanies the book. It is suggested that blackboard protractors, also, may be used to advantage.

Incommensurable quantities and the theory of limits are treated in a rational manner. The consideration of incommensurable numbers is introduced informally in the treatment of areas. The discussion of the theory of limits is postponed until Chapter V, where it is treated in connection with the measurement of the circle.

Some of the features peculiar to the Solid Geometry are outlined in the following paragraphs:

1. In Chapter VI, the theorems on the relations of lines to planes and of planes to planes are carefully grouped so as to make them more easily comprehended by the student.

2. Distinct advantage is gained by combining certain related subjects, thus securing brevity and simplicity of treatment. Prisms are combined with cylinders, pyramids with cones, and polyedral angles with spherical polygons.

3. In the theorems regarding the areas and volumes of solids, and requiring a use of limits, great care is taken to make the treatment logical and still keep it within the grasp of the average student.

4. The Prismatoid Formula, one of the most powerful theorems of solid geometry, is proved in a simple, direct manner, and is followed by various applications. However, this may be omitted and in no way interfere with the proofs of later theorems.

The more important theorems are printed in heavy type. This is in accordance with the recommendation of the Committee of Fifteen of the National Education Association,

whose report has been given careful consideration in the preparation of the entire work.

Some of the ideas in this text and the material for many of the exercises have been obtained through a wide reading of authorities found in various university and technical libraries. The numerous practical applications have been gleaned from a variety of sources covering a period of many years, and pertain to nearly all phases of life.

Acknowledgment is due to the McGraw-Hill Book Co., Inc., for permission to use exercises from Palmer's *Practical Mathematics;* to Prof. James H. Breasted, and to the publishers, for the illustration on page 8, from *A History of Egypt,* copyright, 1905, 1909, published by Charles Scribner's Sons; to Mr. Lawrence C. Irwin of the Joliet Township High School, and to Mr. F. G. Taylor of the public schools of Oregon, Illinois, who taught the material in manuscript form, and who has given advice of great value; to Miss Margaret Willcox of Oak Park, Illinois, who critically read all the proof, and to many students of the Oak Park High School whose response to the study of the manuscript gave encouragement in the preparation of the work.

C. I. PALMER.
D. P. TAYLOR.

Chicago, 1919.

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PRELIMINARY STATEMENT

Any physical body, such as a book, a tree, or a tract of land, has certain qualities that refer to shape, size, and relative position with respect to other bodies. These qualities are known as the *geometric properties* of a body, and the science which deals with them is called geometry. Arithmetic centers about computation with numbers, and algebra about the equation. Geometry makes use of both of these subjects and applies them in determining the form of objects and in comparing their relative position and magnitude.

It is apparent on every hand that geometric principles play an important part in everyday life. Their use is seen in mosaic tiling and parquet flooring, in the designs of stained-glass windows, in the architecture of buildings, in engineering constructions, and in the measurement of land by the surveyor.

GEOMETRY AMONG THE ANCIENTS

The oldest traces of geometry are found among the Egyptians and Babylonians. That a fund of mathematical facts was known to these people at a very early date is evidenced in the construction of the great pyramids, temples, and mausoleums, which could not have been erected without a knowledge of geometric principles. The subject was of special importance to the Egyptians because of the frequent land surveys necessitated by the overflow of the Nile. The inundation of the Nile near the pyramids of Gizeh, is shown in this illustration.

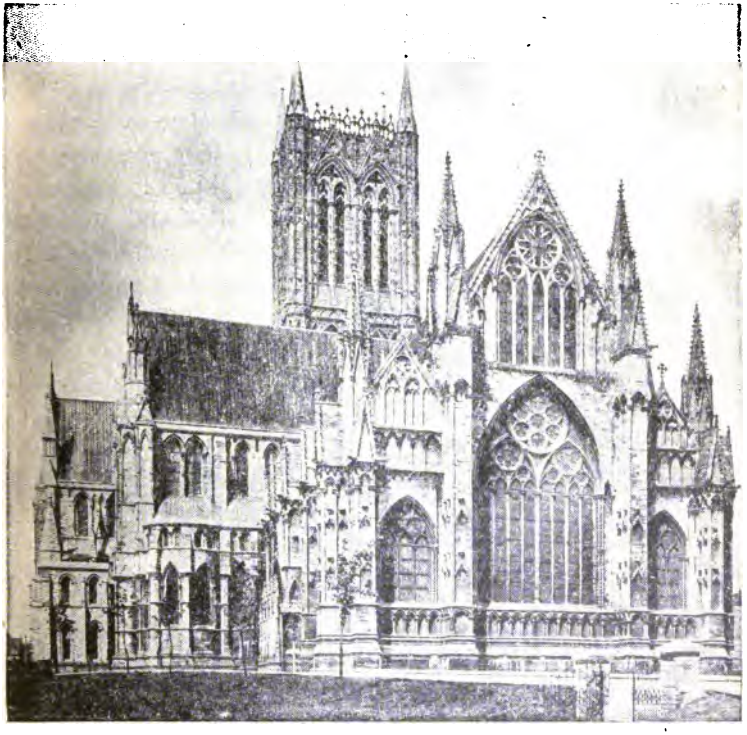
THE BIRTHPLACE OF GEOMETRY

The oldest mathematical work known to exist was written by an Egyptian named Ahmes, who lived about 1700 B.C. This manuscript, which is now in the British Museum, treats of many kinds of algebraic and geometric problems, and mentions still older treatises on mathematics.

As early as the seventh century B.C. the geometry of the Egyptians became known to the Greeks. About 300 B.C., their great teacher of mathematics, Euclid, brought forth a masterpiece known as *Euclid's Elements*, which, until recent years, was almost the universal text book on geometry.

THE USES OF GEOMETRY

In contact with the material world, a person more or less unknowingly gains a great fund of geometric facts. The study of geometry is to call attention to these facts, to lead one to observe other similar facts, to relate and systematize them, and



to use them in solving various interesting as well as practical problems.

Geometry gives assistance in many of the sciences which determine the advance of material civilization. The architect, in planning a building, utilizes geometric principles both to

make the structure secure and to render it beautiful. The illustration of Lincoln Cathedral on the preceding page is an excellent example of this. The constructing engineer is able to design the framework of buildings and bridges capable of withstanding tremendous stress by the use of triangular bracing, as shown in the illustration on page 7. The sailor shapes his course and avoids dangerous localities on a voyage through his knowledge of the laws of geometry. The astronomer and the surveyor base their measurements and calculations upon geometric facts. The machinist relies upon the same laws in laying out his work. The carpenter applies geometry every time he uses his "square." The mechanic, the draftsman, and the electrician make frequent use of the facts established in geometry.

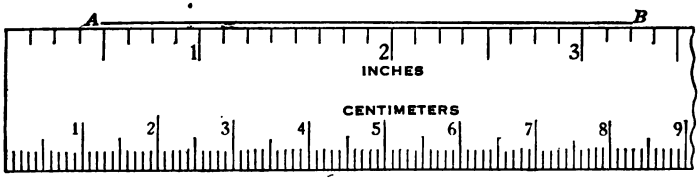
A knowledge of geometry brings pleasure and increased capacity for enjoyment by developing an appreciation for the beauty and utility of architecture, of art, and of engineering constructions. It enables the landscape gardener to develop plans for magnificent parks and public gardens. It also forms the basis of all civic improvement plans for making cities more sanitary and more attractive.

In mature life, persons encounter problems which cannot be solved by rule and are mastered only by resourcefulness and experience. The propositions and exercises of geometry develop self-reliance and a power of initiative because of the opportunities which they present for original thinking and investigation.

Finally, the student receives in geometry the only training in formal logic which is obtained in the secondary school. He gains a knowledge of pure argumentation and of sound reasoning and learns to appreciate the force and value of concise statement in the exact use of language.

PLANE GEOMETRY

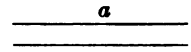
INTRODUCTION



1. Drawing a Straight Line. Place a *ruler* or *straightedge* on a piece of paper and draw a pencil along the edge so that it makes a continuous mark.

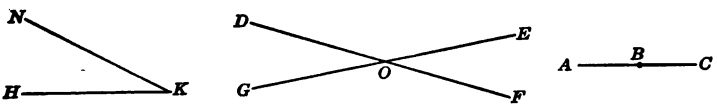
The line AB is a **straight line.**

A line may also be designated by a single letter. The first line is read, "the line *a*." The second line is read, "the line *l*."



In drawing lines, use a hard pencil sharpened to a fine point. A straightedge may be made by folding a piece of paper and creasing the edge.

2. Point. The place where two lines meet or intersect each other is a **point.** The ends of a line are **points.** The place where a line is separated into parts is a **point.**

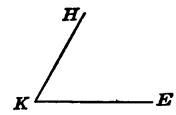


The lines HK and NK meet at the point K. The lines DF and GE intersect at the point O. The ends A and C of the line AC are points. The line AC is separated into the two parts AB and BC by the point B.

A point is represented by a dot. To aid in speaking about a point, a letter, as P, may be placed beside the dot. The point is then read, "the point P."

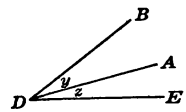
3. Angle. Two straight lines which meet at a point form an **angle.** The two lines are called the **sides** of the angle. The point of meeting is called the **vertex** of the angle.

This angle is read, "the angle *HKE*," "the angle *EKH*," or "the angle *K*." When all three letters are read, the letter at the vertex is read between the other two.

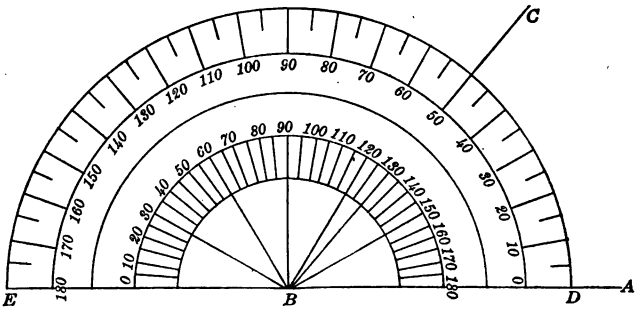


The lines *KH* and *KE* are the sides of the angle and the point *K* is the vertex of the angle.

An angle may also be designated by a letter placed within the angle near the vertex. In this figure, the angle *BDA* may be read, "the angle *y*," and the angle *ADE* may be read, "the angle *z*." The angle *BDE* may also be read, "the angle *D*."

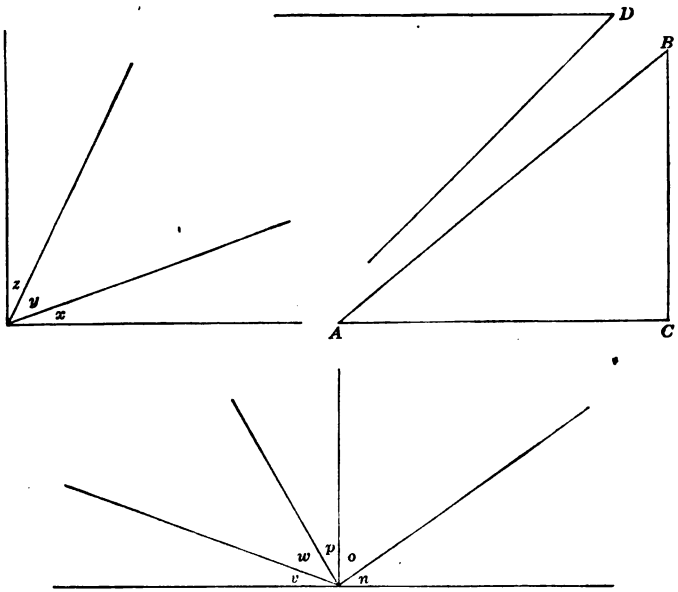


4. Protractor. An instrument for determining the number of degrees in an angle, and also for drawing an angle of a given number of degrees is called a **protractor.**



5. Measuring an Angle. To measure an angle, place the point *B* of the protractor on the vertex of the angle. Make one edge of the protractor, as *BD*, coincide with one side of the angle, as *BA*. The position of the other side *BC* indicates the number of degrees in the angle. The angle *ABC* is an angle of 50°.

Measure each of the following angles with the protractor.



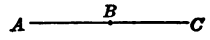
6. Drawing an Angle. To draw an angle, first draw a straight line, which is to be one side of the angle, as *BC* in the figure, § 4. If the angle is to be 50°, place the protractor so that the side *BC* falls along the 50° mark, then draw the side *BA* along the edge. The angle *ABC* is the required angle of 50°.

Care should be taken to draw all angles accurately.

Draw angles of 20°, 45°, 75°, 90°, 125°, 135°, 180°, 35°, 145°, 110°.

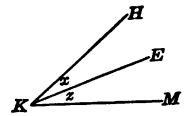
7. Accuracy in Drawing Lines and Angles. In drawing a line through a point, first place the pencil upon the point, then bring the ruler up to the pencil. In drawing a line through two points, try the pencil at each point, moving the ruler until the line passes exactly through both points. In drawing an angle with the vertex at a given point, care should be taken to make the sides meet at that point.

8. Bisector. A figure is **bisected** when it is divided into two *equal* parts.



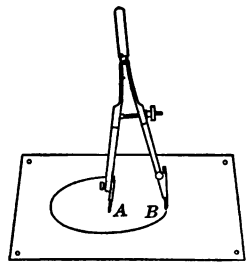
The line AC is bisected by the point B, since it is divided into the two equal parts, AB and BC. The point B is the **bisector** of the line AC.

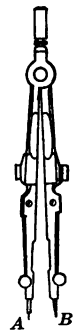
The line KE is the **bisector** of the angle K, since it divides the angle into the two equal angles, x and z.



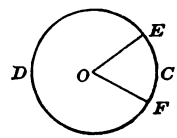
With the protractor, draw angles of 120°, 150°, 30°, 70°, 170°, and 80°. Bisect these angles with the protractor.

9. Compasses. A pair of **compasses** is an instrument for constructing circles and laying off lengths on lines. The compasses can be adjusted for any desired distance between the two points A and B.





10. Circle. Place one point A of the compasses on a point O, and adjust the compasses until the other point B falls on point D. Keeping the point at O stationary, move point B around point O until it returns to point D. The distance between the points A and B remains unchanged as point B moves about point O.

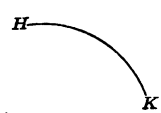


The line DEF is a **curved line** called a **circle.** The point O is the **center** of the circle.

A straight line drawn from the center to any point on the circle is a **radius.**

Thus the line OE is a **radius** of the circle. The lines OE and OF are **radii** of the circle.

Any portion of a circle is an **arc.**

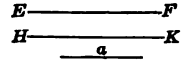


Thus HK is an **arc.** In the circle whose center is O, FCE and FDE are arcs.

Construct ten circles, using a different radius for each.

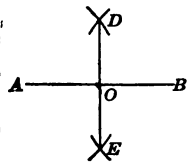
11. Radii of a Circle. Since the distance between the two points of the compasses is not changed during the construction of a circle, *all radii of the same circle are equal.* It follows that, *all points on a circle are equidistant from the center.*

12. Laying off Lengths on Lines. Set the points of the compasses one-half inch apart and lay off one-half inch on the line EF.



Set the points of the compasses on the ends of the line a, then lay off a portion of HK equal to the line a.

13. Bisecting a Straight Line. The line AB is to be bisected. With A and B as centers, and with a radius greater than half of AB, describe arcs which intersect each other at two points D and E. Draw the line DE intersecting AB at O. The line AB is bisected at the point O.

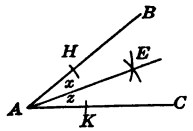


With the aid of compasses, show that AO is equal to OB.

Draw lines one, two, three, and four inches long, respectively. Also draw lines two, four, five, eight, and ten centimeters long, respectively. Bisect these lines by the above method. Test with the ruler or compasses to see if the work is accurate.

Proof of this construction may be found in § 314.

14. Bisecting an Angle. The angle BAC is to be bisected. With A as a center and any convenient radius, describe arcs intersecting the sides of the angle at H and K. With H as center and with a radius sufficiently great, describe an arc; with K as center and the same radius, describe an arc intersecting the first arc at E. Draw the line AE. The angle BAC is bisected by the line AE, making angle x equal to angle z.



With the protractor, draw angles of 50°, 70°, 90°, 120°, 130°, and 150°. Bisect these angles by the above method. Test with the protractor to see if the work is accurate.

Proof of this construction may be found in § 319.

SYMBOLS AND ABBREVIATIONS

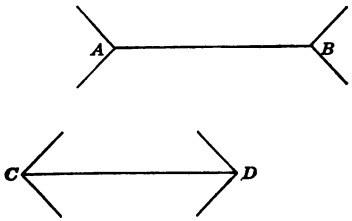
$=$	equals, is equal to, or is equivalent to.	$\odot$	circle.
$>$	is greater than.	ⓢ	circles.
$<$	is less than.	$\overset{\frown}{AB}$	arc AB.
$\backsim$	is similar to.	$=^\circ$	equals in degrees.
$\therefore$	therefore.	$\rightarrow$	approaches as a limit.
$\angle$	angle.	$\cong$	is congruent to.
$\measuredangle$	angles.	cm.	centimeter.
$\triangle$	triangle.	rt.	right.
$\triangle\!\!\triangle$	triangles.	st.	straight.
$\perp$	perpendicular, or is perpendicular to.	Def.	definition.
		Ax.	axiom.
		Cons.	construction.
$\parallel$	is parallel to.	Adj.	adjacent.
$\square$	parallelogram.	Iden.	identical.
ⓢ	parallelograms.	Ex.	exercise.
$\square$	rectangle.	Sup.	supplementary.
ⓢ	rectangles.	Ext.	exterior.
$\square$	square.	Int.	interior.
ⓢ	squares.	Alt.	alternate.

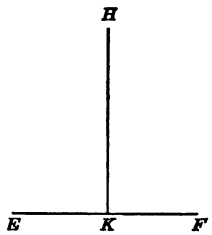
CHAPTER I

STRAIGHT-LINE FIGURES

15. Optical Illusions. Is AB equal to CD, longer than CD, or shorter than CD? Estimate first by sight, only. Test with compasses or ruler.

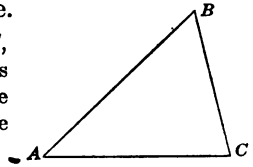
Is HK equal to EF, longer than EF, or shorter than EF? Can the eye be trusted in comparing the size of objects?





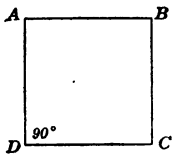
16. Triangle. If a portion of surface is bounded by three straight lines, a **triangle** is formed. The bounding lines are called the **sides** of the triangle. The three angles formed by the sides are called the **angles** of the triangle. The vertices of the three angles are the **vertices** of the triangle.

Thus ABC is a triangle. The lines AB, BC, and CA are the sides of the triangle. The angles A, B, and C are the angles of the triangle. The points A, B, and C are the vertices of the triangle.



17. Square. If a portion of surface is bounded by four straight lines, so that all the sides are equal and all the angles are equal, a **square** is formed.

Thus $ABCD$ is a square, since the sides AB, BC, CD, and DA are all equal, and the angles A, B, C, and D are all equal.



18. Experiment. Cut out of heavy paper a triangle ABC with the same dimensions as Fig. 1, making $AC=4$ cm, $\angle A=40°$, $\angle C=60°$. Place this triangle on Fig. 2 so that AC coincides with DF. The triangles ABC and DEF have a side of one equal to a side of the other. Are these two triangles equal?

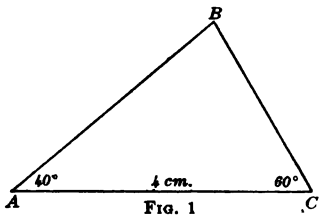


FIG. 1

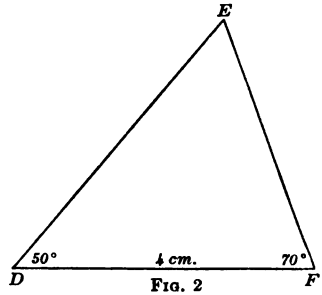


FIG. 2

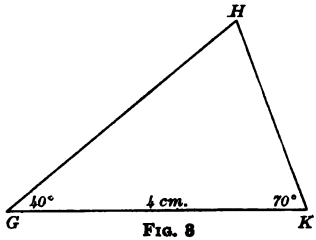


FIG. 3

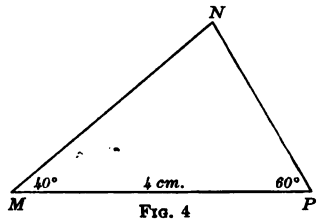


FIG. 4

Place the triangle ABC on Fig. 3 so that AC coincides with GK. Triangles ABC and GHK have a side and an angle of one equal respectively to a side and an angle of the other. Are these two triangles equal?

Place the triangle ABC on Fig. 4 so that AC coincides with MP. Triangles ABC and MNP have two angles and the included side of one equal respectively to two angles and the included side of the other. Are these two triangles equal?

The following expresses the result of this experiment:

19. ***If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are equal.***

Proof of this statement may be found in § 248.

20. By means of the ruler, compare the length of the line AB with the lengths of the lines DE, GH, and MN, § 18. In the same way, compare the line BC with the lines EF, HK, and NP.

By means of the protractor, compare the number of degrees in $\angle B$ with the number of degrees in $\measuredangle E$, H, and N.

Since $\triangle ABC = \triangle MNP$, it must also be true that $AB = MN$, $BC = NP$, and $\angle B = \angle N$. Therefore it can be said that:

21. ***In equal triangles, the corresponding sides are equal and the corresponding angles are equal.***

EXERCISES

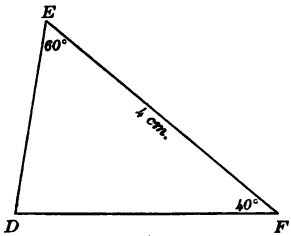


FIG. 5

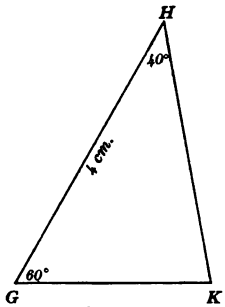
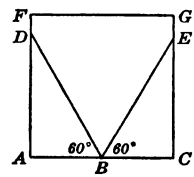


FIG. 6

1. Place the triangle which was cut out equal to Fig. 1, § 18, so that AC coincides with EF of Fig. 5. Then place the same triangle so that AC coincides with GH of Fig. 6. Is it equal to each of these triangles? Does the position of a triangle affect its size?

2. The figure $AFGC$ is a square. B is the middle point of AC. The lines DB and EB are drawn making the angles ABD and CBE each equal to 60°. Prove that $\triangle BDA = \triangle BEC$.



Proof. $\angle A = \angle C$. § 17

(*Since the angles of a square are equal.*)

$AB = BC$ Given

(*By the statement that B is the middle point of AC.*)

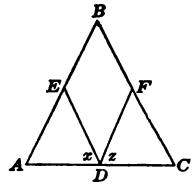
$\angle ABD = \angle CBE$.

(*Each being 60°.*)

$\therefore \triangle BDA = \triangle BEC$. § 19

(*If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are equal.*)

3. The angles A and C of the triangle ABC are equal. The point D bisects the side AC. The lines DE and DF are drawn making $\angle x = \angle z$. Prove that $\triangle ADE = \triangle CDF$.



4. In $\triangle ABC$, $\angle A = \angle C$. The line DB bisects $\angle ABC$. $AB = BC$. Prove that $AD = DC$.

Proof.

$\angle A = \angle C$. Given

$\angle x = \angle y$.

(*Since DB bisects* $\angle ABC$.)

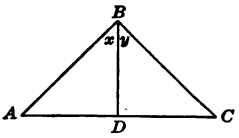
$AB = BC$. Given

Then $\triangle ABD = \triangle CBD$.

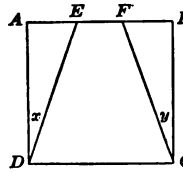
(*Two angles and the included side are respectively equal.*)

$\therefore AD = DC$.

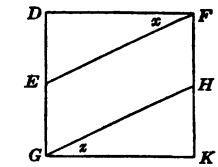
(*In equal triangles, the corresponding sides are equal.*)



5. The figure $ABCD$ is a square. The lines DE and CF are drawn, making angles of 20° with AD and BC respectively. Prove that $DE = CF$.



6. In the square $GDFK$, $\angle x$ and $\angle z$ are each 25°. Prove that $GH = EF$.



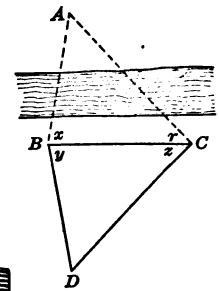
***7.** Show how the distance from a point B to an inaccessible point A may be found, as shown in the figure.

Any distance as BC is laid off from B. $\angle x$ is found by observation and $\angle y$ is made equal to $\angle x$. Similarly $\angle z$ is made equal to $\angle r$.

$\triangle DBC = \triangle ABC$. Why?

$\therefore BD = BA$. Why?

Then the distance from B to A is determined by measuring BD.

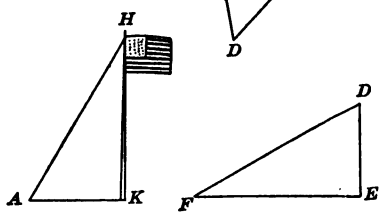


***8.** The height of a flag pole may be found by laying off on the ground a triangle DFE which is equal to triangle AHK.

To do this, lay off a line $DE = AK$. Then make at D an angle equal to $\angle A$ and at E an angle equal to $\angle K$.

Why does $\triangle AHK = \triangle DFE$?

Why is EF equal to KH, the height of the pole?



22. Experiment. Cut out a triangle ABC with the same dimensions as Fig. 1, making $AC = 4$ cm., $AB = 3$ cm., $\angle A = 40°$.

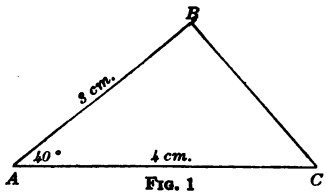


Fig. 1

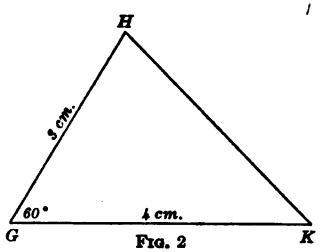


Fig. 2

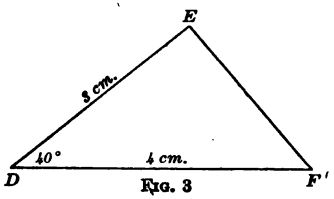


Fig. 3

Place the triangle ABC on Fig. 2 so that AC coincides with GK. Triangles ABC and GHK have two sides of one equal respectively to two sides of the other. Are these two triangles equal?

Place the triangle ABC on Fig. 3 so that AC coincides with DF. Triangles ABC and DEF have two sides and the included angle of one equal respectively to two sides and the included angle of the other. Are these two triangles equal?

The following statement expresses the result of this experiment:

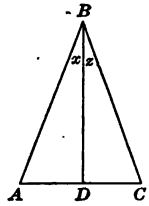
23. ***If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are equal.***

Proof of this statement may be found in § 247.

The three fundamental tests for equality of triangles are comprised in § 19, § 23, and § 27.

EXERCISES

1. In $\triangle ABC$, the side AB equals the side CB, and BD bisects $\angle ABC$. Prove that $\triangle ADB = \triangle CDB$.



Proof.

$\angle x = \angle z$.	
(*Since BD bisects $\angle ABC$.*)	
$AB = CB$.	Given
$BD = BD$.	Iden.
(*Being common to both triangles.*)	
$\therefore \triangle ADB = \triangle CDB$.	

(*Two sides and the included angle are respectively equal.*)

2. In $\triangle ACE$, $\angle A = \angle E$. F is the middle point of AE. $AB = ED$. Prove that $BF = DF$. (Fig. 1.)

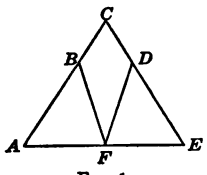


FIG. 1

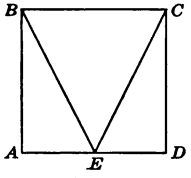


FIG. 2

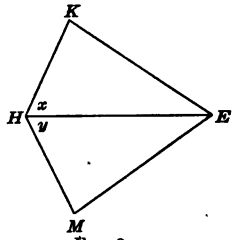
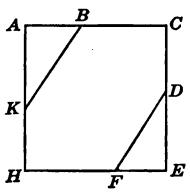


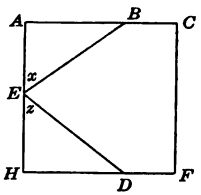
FIG. 3

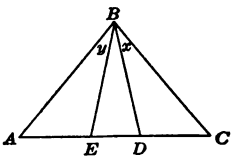
3. In Fig. 2, $ABCD$ is a square. The side AD is bisected at E. Prove that $\triangle ABE = \triangle DCE$.

4. In Fig. 3, HE bisects $\angle KHM$. $HK = HM$. Prove that $KE = ME$.

5. In the square $ACEH$, $AB = EF$ and $AK = ED$. Prove that $\triangle ABK = \triangle EFD$.







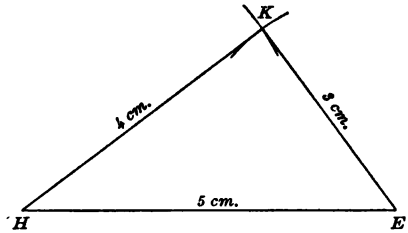
6. In the square $ACFH$, E is the middle point of AH. $AB = HD$. Prove that $\angle x = \angle z$.

SUGGESTION. Show that $\triangle ABE = \triangle HDE$.	§ 23
$\therefore \angle x = \angle z$.	§ 21

(*Corresponding angles of equal triangles are equal.*)

7. In $\triangle ABC$, $AB = CB$ and $\angle A = \angle C$. E and D are equally distant from A and C respectively. Prove that $\angle y = \angle x$.

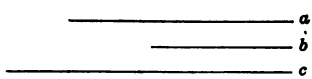
24. Construction. Draw the line $HE = 5$ cm. With H as a center, and a radius of 4 cm., describe an arc. With E as a



center and a radius of 3 cm., describe an arc intersecting the first arc at K. Draw KH and KE. $\triangle HKE$ is constructed with its sides equal to 5 cm., 4 cm., and 3 cm., respectively.

25. *To construct a triangle when the three sides are given.*

Construct a triangle with its sides equal respectively to the three given lines a, b, and c.



26. Experiment. Following the method of § 24, construct and cut out a triangle ABC with the same dimensions as Fig. 1, making $AC = 4$ cm., $AB = 3$ cm., $BC = 2$ cm.

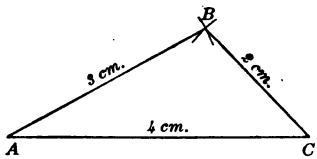


FIG. 1

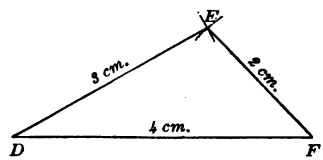


FIG. 2

Place $\triangle ABC$ upon $\triangle DEF$ so that AC coincides with DF. Triangles ABC and DEF have the three sides of one equal respectively to the three sides of the other. Are these two triangles equal? Then the following is true:

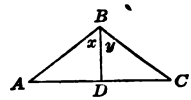
27. ***If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are equal.***

Proof of this statement may be found in § 113.

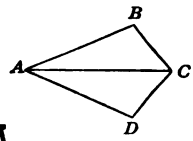
EXERCISES

1. Construct a triangle equal to a given triangle.

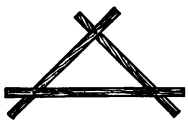
2. In $\triangle ABC$, $AB = BC$ and BD bisects line AC. Prove that $\angle A = \angle C$ and that BD bisects $\angle ABC$.

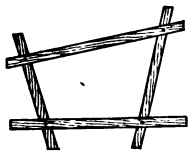


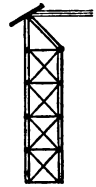
3. In the figure, $AB = AD$ and $BC = DC$. Prove that $\triangle ABC = \triangle ADC$.



4. Why will the framework of three sticks fastened with one nail at each joint hold its shape?







5. Will the framework of four sticks with one nail at each joint hold its shape? How can a fifth stick be placed to make the frame firm?

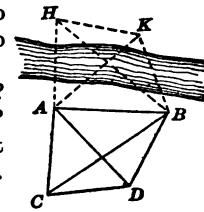
6. Why does the brace AB cause the step ladder to stand solid?

7. In the view of the support used in wireless telegraphy, the braces form triangles. Why is the structure sufficiently braced by this method?

8. Show how the water tower is made to resist severe strain.

***9.** In the framework of buildings and bridges, much use is made of the triangle, as the "unit of rigidity." Why is it possible for this bridge to be supported on a pier at the center?

*10. Show how to find the distance between two points H and K both of which are inaccessible to the observer.



Lay off $\triangle ABC = \triangle ABH$. Then $AC = AH$. Why?

Lay off $\triangle ABD = \triangle ABK$. Then $BD = BK$. Why?

If $AHKB$ were folded over AB, why would it fall on $ACDB$? Hence HK would fall on CD.

$\therefore CD = HK$.

ALGEBRAIC EXERCISES

Find the value of $\angle x$ in each of the following equations.

1. $\angle x = 40° + 55° + 15°$.
2. $\angle x - 20° = 75°$.
3. $\angle x + 5\angle x = 180°$.
4. $25° + 2\angle x = \angle x + 55°$.
5. $5\angle x - 30° - \angle x = 2\angle x + 70°$.
6. $\angle x + \frac{1}{2}\angle x = 50° + 70°$.
7. $\angle x - 19° = 3\angle x - 143°$.
8. $7\angle x - 67°\ 30' = 5\angle x + 22°\ 30'$.
9. $43°\ 20' - 2\angle x = 57°\ 40' - 3\angle x + 26°\ 50'$.
10. $\angle x = 15°\ 29'\ 45'' + 54°\ 27'\ 48'' - 22°\ 15'\ 37''$.
11. $\angle x - 2\angle a = 3\angle a - 3\angle x + 7\angle a$.
12. $\angle x - 20° = 12° - \frac{1}{2}\angle x$.
13. $\frac{1}{2}\angle x + 5° = 10° + \frac{1}{4}\angle x$.
14. $\angle c + \frac{3}{4}\angle x = 7\angle c$.
15. $\angle a + \angle x + \angle c = 180°$.

Find the values of $\angle x$ and $\angle y$ in the following:

16. $\begin{cases} \angle x + 3\angle y = 80°, \\ \angle x - 2\angle y = 30°. \end{cases}$
17. $\begin{cases} \angle x - \angle y = 10°, \\ 4\angle x + \angle y = 90°. \end{cases}$
18. $\begin{cases} \angle x + \angle y = 15°, \\ 2\angle x + 3\angle y = 40°. \end{cases}$
19. $\begin{cases} 5\angle x - 3\angle y = 45°, \\ 10\angle x + 7\angle y = 155°. \end{cases}$
20. $\begin{cases} 5\angle x + 7\angle y = 55°, \\ 9\angle x - 3\angle y = 21°. \end{cases}$
21. $\begin{cases} \angle x - 50° = 2\angle y, \\ 11\angle y - 40° = 2\angle x. \end{cases}$
22. $\begin{cases} \angle x + 2\angle y = 95°\ 45', \\ \angle x - \angle y = 20°\ 15'. \end{cases}$
23. $\begin{cases} 3\angle x + 4\angle y = 93°\ 32', \\ 5\angle x - \angle y = 36°\ 48'. \end{cases}$

Find the values of $\angle x$, $\angle y$, and $\angle z$ in the following:

24. $\begin{cases} \angle x + \angle y + \angle z = 55°, \\ \angle x + 2\angle y + 3\angle z = 115°, \\ 3\angle x + \angle y - 2\angle z = 35°. \end{cases}$
25. $\begin{cases} \angle x + \angle y = 50°, \\ \angle x - \angle z = 20°, \\ \angle y - \angle z = 10°. \end{cases}$

26. If the sum of two angles is 145° and their difference is 35°, how many degrees in each?

27. The sum of the half, third, fifth, and sixth parts of an angle is 18° larger than the angle itself. How many degrees in the angle?

---

60 seconds = 1 minute. $60'' = 1'$.

60 minutes = 1 degree. $60' = 1°$.

28. Experiment. The two triangles ABC and GHK have an angle of one equal to an angle of the other. Are these two triangles equal?

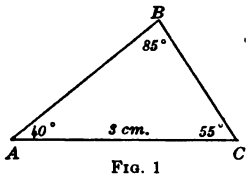


FIG. 1

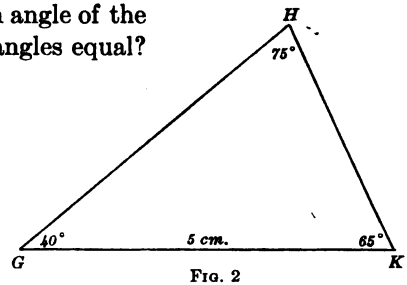


FIG. 2

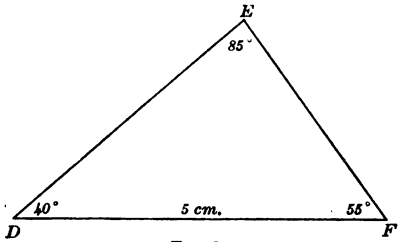


FIG. 3

The two triangles ABC and DEF have the three angles of one equal respectively to the three angles of the other. Are these two triangles equal?

Are two triangles *necessarily* equal when the three angles of one are equal respectively to the three angles of the other?

29. Experiment. What is the sum of all the angles of each triangle in § 28? In the four figures of § 18, find with the protractor the number of degrees in $\measuredangle B$, E, H, and N. What is the sum of the three angles of each triangle? It follows that:

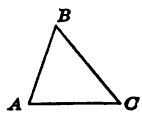
30. ***The sum of the angles of a triangle is equal to 180°.***

Proof of this statement may be found in § 143.

31. From § 30, $\angle A+\angle B+\angle C=180°$.

$$\therefore \angle C=180°-\angle A-\angle B.$$

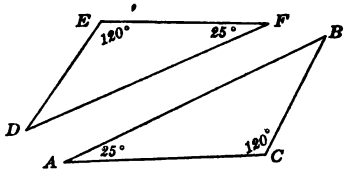
Or $$\angle C=180°-(\angle A+\angle B).$$



If two angles of a triangle are known, the third may be found by subtracting their sum from 180°.

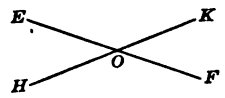
32. How many degrees in $\angle D$? In $\angle B$? § 31

If two triangles have two angles of one equal respectively to two angles of the other, the third angles are equal.



33. Vertical Angles. If the sides of one angle are the prolongations of the sides of another angle, the angles are called **vertical angles.**

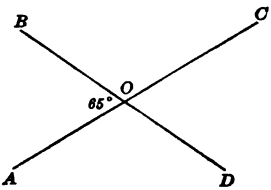
Thus $\angle EOH$ and $\angle KOF$ are vertical angles. $\angle EOK$ and $\angle HOF$ also are vertical angles.



34. Experiment. To construct this figure, first draw the line AC. Then draw the line BD, making $\angle AOB = 65°$. With the protractor, find the number of degrees in each of the other angles.

How do $\measuredangle AOB$ and DOC compare?

How do $\measuredangle AOD$ and BOC compare?



35. ***If one straight line intersects another straight line, the vertical angles are equal.***

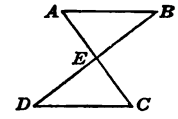
Proof of this statement may be found in § 56.

EXERCISES

When $\angle A$ and $\angle B$ of a triangle ABC have the following values, find the value of $\angle C$:

1. $\angle A = 60°$, $\angle B = 70°$, $\angle C = ?$
2. $\angle A = 40°$, $\angle B = 80°$, $\angle C = ?$
3. $\angle A = 20°$, $\angle B = 35°$, $\angle C = ?$
4. $\angle A = 50°$, $\angle B = 85°$, $\angle C = ?$
5. $\angle A = 75°$, $\angle B = 65°$, $\angle C = ?$
6. $\angle A = 42°$, $\angle B = 31°$, $\angle C = ?$
7. $\angle A = 59°$, $\angle B = 84°$, $\angle C = ?$
8. $\angle A = 48°$, $\angle B = 97°$, $\angle C = ?$
9. $\angle A = 21° 34'$, $\angle B = 85° 27'$, $\angle C = ?$
10. $\angle A = 47° 56' 34''$, $\angle B = 72° 46' 26''$, $\angle C = ?$
11. Construct with ruler and protractor the first four triangles given above, making the side AB two inches long. Test $\angle C$ with the protractor and compare with the results obtained by computation.

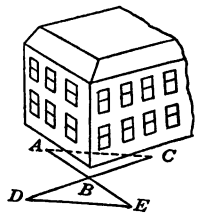
12. The point E bisects the line AC and the line BD. Prove that $\triangle ABE = \triangle CDE$.



13. The point E bisects the line BD and $\angle B = \angle D$. Prove that the point E also bisects the line AC.

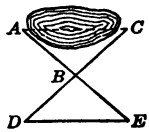
14. Find the distance from a point A, west of a building, to a point C, south of the building, if the building prevents direct measurement.

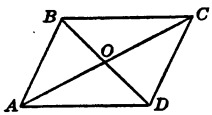
A point B is taken. CB is produced to D, making $BD = CB$. AB is produced to E, making $BE = AB$. Why does $DE = AC$?

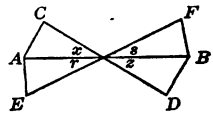


15. Draw a triangle, DEF, making $\angle D = \angle F$. Locate any point H in the line DF. Locate a point A in DE, so that $DA = HF$. Also locate a point B in FE, so that $FB = DH$. Draw HA and HB and prove that $HA = HB$.

16. Show that DE is equal to the distance across the lake from A to C.







17. The lines AC and BD bisect each other at O. Prove that $\triangle AOB = \triangle COD$ and $\triangle BCO = \triangle DAO$.

18. The lines AB, CD, and EF bisect each other. Prove that $AC = BD$ and $AE = BF$.

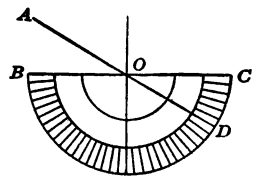
***19.** Show how the angle of elevation of the sun may be found with a protractor. If the sun is at A, the $\angle AOB$ is called the *angle of elevation* of the sun. The line AO from the sun runs to D.

$\angle DOC = \angle AOB$. Why?

How many degrees in $\angle DOC$?

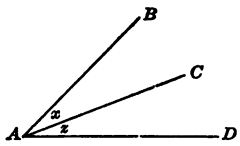
Then what is the angle of elevation of the sun?

The line AO may be projected to D by using a small piece of paper and observing the shadow.



FUNDAMENTAL PRINCIPLES

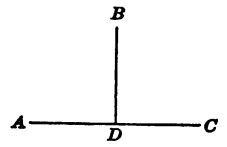
36. Adjacent Angles. Two angles which have the same vertex and a common side between them are **adjacent angles.**



Angles x and z are adjacent angles, since they have the same vertex A and the common side AC between them.

Angle BAD is the *sum* of the two adjacent angles, x and z. $\angle z$ is the *difference* between $\angle BAD$ and $\angle x$. What is the *difference* between $\angle BAD$ and $\angle z$?

37. Right Angle. When one straight line meets another straight line, making the adjacent angles equal, each of these angles is a **right angle.**



Angles ADB and CDB are right angles, since BD meets AC, making $\angle ADB = \angle CDB$.

Examine the protractor and find how many degrees there are in a right angle. How can a right angle be drawn with a protractor?

38. Perpendicular. A straight line which makes a right angle with another straight line is **perpendicular** to it.

Thus BD is perpendicular to AC. AC is also perpendicular to BD.

How can a line be drawn perpendicular to another line with a protractor?

The point D is called the **foot** of the perpendicular BD.

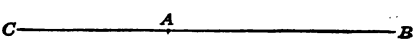
39. Vertical Line. A **vertical line,** or **plumb line,** is the line along which a string hangs when suspended from one end and weighted at the other.

Thus BD, in the figure of § 37, is a vertical line.

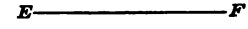
40. Horizontal Line. A **horizontal line** is a line that is perpendicular to a vertical line.

Thus AC, in the figure of § 37, is a horizontal line.

41. Direction. A point A is taken. AB is laid off 3 cm. to the right of A, and AC is laid off 2 cm. to the left of A, in the same straight line with AB. Then AB and AC extend in *opposite directions* from the point A.



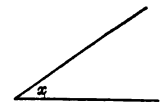
The line EF may be considered as extending from *left to right* in the direction EF, or it may be considered as extending from *right to left* in the direction FE.



42. Straight Angle. If the sides of an angle extend in opposite directions so as to form a straight line, the angle is a **straight angle.**

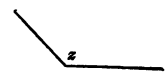
Angle BAC is a straight angle. Examine the protractor and find how many degrees there are in a straight angle.

43. Acute Angle. An angle which is less than a right angle is an **acute angle.**



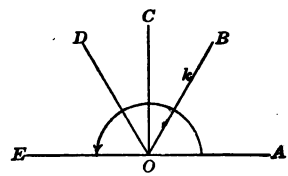
Angle x is an acute angle.

44. Obtuse Angle. An angle which is greater than a right angle, but less than a straight angle, is an **obtuse angle.**



Angle z is an obtuse angle.

45. Generation of Angles. If one side of an angle moves about the vertex away from the other side, the angle increases in size. If the line k at first coincides with OA, and then moves about O as a vertex to the position OB, the acute angle AOB is formed or **generated.** If the line k then moves about O until it becomes perpendicular to OA, it takes the position OC and the right angle AOC is formed. If the line k moves to the position OD, the obtuse angle AOD is formed. If the line k takes the position OE in the same straight line with OA, the straight angle AOE is formed.

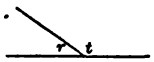


If OA is considered as extending to the right and if k has the same direction when it coincides with OA, then the direction of k may be considered outward from O whatever its position. This gives OE the direction opposite to that of OA in the straight angle AOE.

46. Complementary Angles. Two angles whose sum is a right angle are **complementary angles,** and each angle is the **complement** of the other.

Angles x and z are complementary. $\angle z$ is the complement of $\angle x$. What is the complement of $\angle z$?

47. Supplementary Angles. Two angles whose sum is a straight angle are **supplementary angles,** and each angle is the **supplement** of the other.



Angles r and t are supplementary. What is the supplement of $\angle r$? Of $\angle t$?

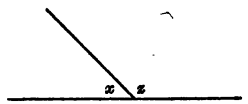
EXERCISES

1. How many degrees in each of the following fractions of a right angle: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18}, \frac{1}{30}, \frac{1}{45}, \frac{2}{3}, \frac{2}{5}, \frac{2}{9}, \frac{2}{15}, \frac{2}{45}, \frac{3}{5}, \frac{4}{5}, \frac{5}{6}$?

2. How many degrees in each of the following fractions of a straight angle: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{12}, \frac{1}{15}, \frac{1}{18}, \frac{1}{20}, \frac{1}{30}, \frac{1}{36}, \frac{1}{45}, \frac{1}{90}, \frac{2}{3}, \frac{3}{4}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{6}$?

3. A right angle is divided into two parts and one of them is 50°. How many degrees in the other part? If one part is 25°, how many degrees in the other? If one part is 47° 30′, how large is the other? If one part is x degrees, how large is the other? In § 46, $\angle z = 20°$. How many degrees in $\angle x$?

4. In the figure, $\angle x$ is 40°. How many degrees in $\angle z$? In § 47, $\angle t = 145°$. How many degrees in $\angle r$?



5. If two angles are supplementary and one is 100°, how many degrees in the other? If one of two supplementary angles is 75°, how large is the other? If one is 112° 30′, how large is the other? If one is x degrees, what is the other?

6. Find the complement of an angle of 35°; of 85°; of 15°; of 63°; of 29°; of 47°; of 27° 30′; of 38° 25′; of 56° 29′ 48″; of 15° 30′ 26″.

7. Find the supplement of an angle of 45°; of 165°; of 70°; of 25°; of 115°; of 136°; of 57°; of 120°; of 31°; of 49° 45′; of 27° 52′; of 63° 43′ 29″; of 144° 27′ 16″.

8. Draw an acute angle. Construct its complement and its supplement. Show that the difference between the complement and the supplement of the angle is a right angle.

9. The complement of an angle has twice as many degrees as the angle itself. How many degrees in each?

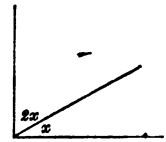
Let x = number of degrees in the angle.

$2x$ = number of degrees in the complement.

$3x = 90$.

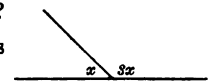
$x = ?$

$2x = ?$



10. How many degrees in an angle whose complement is five times as large as the angle itself? n times as large as the angle?

11. An angle has a supplement three times as great as itself. How many degrees in the angle?



12. How many degrees in an angle whose supplement is four times as great as the angle itself?

13. If one of two complementary angles is 10° larger than the other, how many degrees in each?

14. If the difference between two complementary angles is 20°, how many degrees in each?

15. How many degrees in the angle formed by the hands of a clock at 2 o'clock? At 3 o'clock? At 5 o'clock? At 6 o'clock?

16. How many degrees in the angle formed by the hands of a clock at 2:24 o'clock? At 3:48 o'clock? At 9:12 o'clock?

SUGGESTION. The minute hand moves through 6° each minute. The hour hand moves through 6° in 12 minutes.

17. How many degrees in an angle which is one-fifth of its supplement?

18. How many degrees in an angle which is one-fourth of its complement?

19. How many degrees in an angle when the complement of the angle is equal to one-third of its supplement?

20. How many degrees in an angle when the sum of its complement and its supplement is equal to 170°?

21. What kind of angle is the supplement of an acute angle? Of a right angle? Of an obtuse angle?

22. What kind of angle is formed by the sum of a right angle and an acute angle? By the difference? Illustrate each.

23. With the protractor, construct two angles which are: (1) both supplementary and adjacent; (2) supplementary but not adjacent; (3) adjacent but not supplementary.

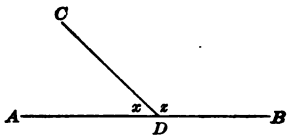
24. What kind of angle is greater than its supplement?

25. If $\angle x = 80°$, $\angle y = 50°$, $\angle z = 160°$, $\angle r = 25°$, $\angle t = 70°$:

(1) $\angle x + \angle y = ?$ (2) $\angle z - \angle y = ?$ (3) $2\angle r + \frac{\angle z}{8} = ?$ (4) $\frac{\angle z - \angle t}{2} = ?$

(5) $\frac{\angle x - \angle r + \angle y}{5} = ?$ (6) $\frac{\angle z}{4} + \angle r - \frac{\angle x}{5} = ?$ (7) $3\angle r + 2\angle y = ?$

48. The two adjacent angles, x and z, have their exterior sides, DA and DB, in the same straight line AB.



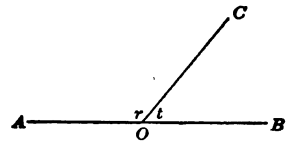
Then $\angle x + \angle z = \text{st.}\angle ADB$.

$\therefore$ $\measuredangle$ x and z are supplementary. § 47

Therefore it is true that:

49. *If two adjacent angles have their exterior sides in the same straight line, they are supplementary.*

50. The two adjacent angles, r and t, are supplementary.



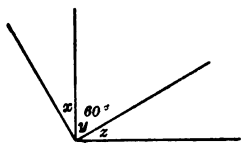
Then $\angle AOB$ is a straight angle, since it is the sum of the two supplementary angles, r and t. § 47

$\therefore$ AO and OB are in the same straight line. § 42

Therefore it is true that:

51. *If two adjacent angles are supplementary, their exterior sides are in the same straight line.*

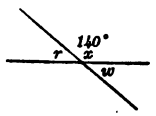
52. $\angle y = 60°$. $\angle x$ is the complement of $\angle y$. How many degrees in $\angle x$? $\angle z$ is also the complement of $\angle y$. How many degrees in $\angle z$?



Then it may be concluded that:

53. *Complements of the same angle or of equal angles are equal.*

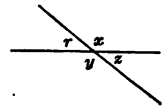
54. $\angle x = 140°$. $\angle r$ is the supplement of $\angle x$. How many degrees in $\angle r$? $\angle w$ is the supplement of $\angle x$. How many degrees in $\angle w$? Compare $\angle r$ and $\angle w$.



Then it may be concluded that:

55. *Supplements of the same angle or of equal angles are equal.*

56. Without using the protractor, prove that *vertical angles are equal.*



Given the vertical angles r and z.

To prove $\angle r = \angle z$.

Proof. $\angle r$ is the supplement of $\angle x$. § 47

$\angle z$ is the supplement of $\angle x$.

$\therefore \angle r = \angle z$. § 55

In a similar manner, prove that $\angle x = \angle y$.

EXERCISES

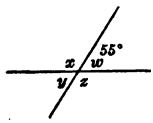


FIG. 1

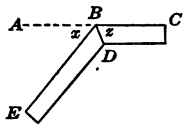


FIG. 2

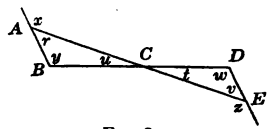


FIG. 3

1. In Fig. 1, angle w is 55°. Find the number of degrees in each of the angles x, y, and z, without using the protractor.

2. Two timbers, Fig. 2, are joined so that the joint BD makes equal angles with the edges of the timbers.

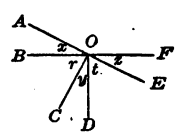
EB meets BC so that $\angle x = 50°$. How large is $\angle z$?

3. In Fig. 3, point C bisects line BD and $\angle y = \angle w$. Prove that $\angle x = \angle z$.

Show that $\triangle ABC = \triangle EDC$. § 19

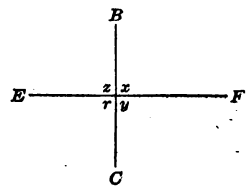
Then $\angle r = \angle v$. § 21

$\therefore \angle x = \angle z$. § 55



4. Lines BF and AE intersect at O. $CO \perp AE$ and $DO \perp BF$. Prove that $\measuredangle x$, y, and z are equal.

57. With the protractor, draw the line BC intersecting the line EF, making $\angle x = 90°$. How many degrees in $\angle z$? Why? In $\angle r$? Why? In $\angle y$? Why?



58. *If one of the angles formed by two intersecting lines is a right angle, the other three angles also are right angles.*

59. *The sum of all the angles about a point is equal to 360°, or four right angles.*

60. *The sum of all the angles about a point, on the same side of a straight line passing through the point, is equal to 180°.*

EXERCISES

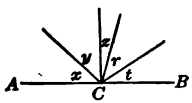


FIG. 1

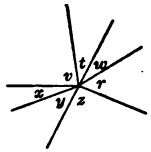


FIG. 2

FIG. 3

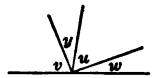


FIG. 4

1. How many degrees in $\angle x+\angle y+\angle z+\angle r+\angle t$? (Fig. 1.) § 60
In $\angle v+\angle t+\angle w+\angle r+\angle z+\angle y+\angle x$? (Fig. 2.) § 59
In $\angle y+\angle x+\angle v+\angle t$? In $\angle z+\angle r+\angle w$? (Fig. 2.) § 60
Test results with the protractor.

2. In Fig. 3, $\angle z=3\angle x$, $\angle r=6\angle x$, $\angle t=2\angle x$. How many degrees in each angle?

3. In Fig. 4, $\angle u=3\angle w$, $\angle y=\frac{3\angle w}{2}$, $\angle v=\frac{7\angle w}{2}$. How many degrees in each angle?

4. In Fig. 5, construct $\measuredangle x$, y, and z with their vertices at the point A, making $\angle z=2\angle x$ and $\angle y=3\angle x$, when it is true that $\angle x+\angle y+\angle z=180°$.

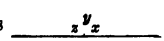
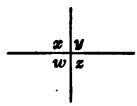


FIG. 5

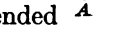
61. From the definition of a right angle, § 37, it follows that *all right angles are equal.*

62. From the definition of a straight angle, § 42, it follows that *all straight angles are equal.*



63. Since there are 90° in a right angle and 180° in a straight angle, *a right angle is half a straight angle.*

64. With the aid of a ruler, connect two points, as A and B, by a straight line. Try to draw another straight line joining the two points A and B. What is your conclusion? How far can line AB be extended beyond the point A? How far can it be extended beyond the point B?

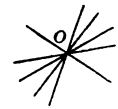


65. *One straight line, and only one, can be drawn from one point to another.*

66. *Two points determine a straight line.*

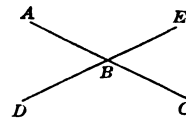
67. *A straight line may be produced indefinitely in both directions.*

68. From the figure, what may be concluded as to the number of straight lines which can be drawn through the point O?



Any number of straight lines can be drawn through a given point.

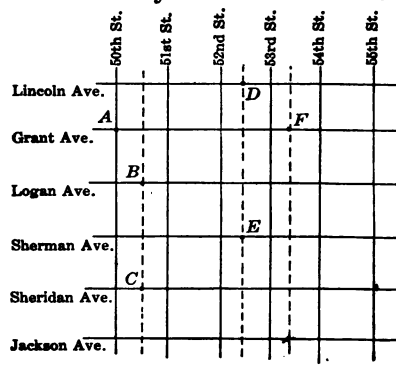
69. The two lines AC and DE intersect in the point B. With a straightedge try to draw two lines which will intersect in more than one point. What is your conclusion?



70. *Two straight lines can intersect in only one point;* for if they had two points in common, they would coincide and form one straight line. § 66

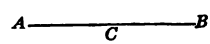
71. *Two intersecting lines determine a point.* Why is the point H appropriately represented by the cross lines?

72. Points on maps are located by latitude and longitude. The location is determined by the intersection of parallels and meridians. Points in cities are located by street and number. The street and number denote two lines intersecting at a point, as in the following diagram:



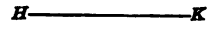
A. 5000 Grant Ave.
B. 5056 Logan Ave.
C. 5056 Sheridan Ave.
D. 5241 Lincoln Ave.
E. 5241 Sherman Ave.
F. 5335 Grant Ave.
Locate 5335 Jackson Ave.
Locate 5500 Sheridan Ave.

73. With the protractor, draw a line perpendicular to a line AB at a point C in the line. Try to draw another line perpendicular to AB at the point C. What is your conclusion?



74. *At a given point in a given straight line, only one perpendicular can be drawn to that line.*

75. With the protractor, draw a line perpendicular to a line HK from a point O outside the line. Try to draw another line perpendicular to HK from the point O. What do you find?



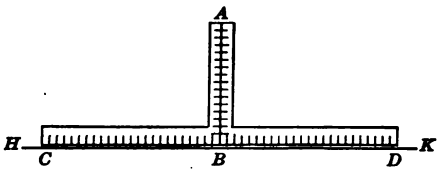
76. *Only one perpendicular can be drawn to a given straight line from a given external point.*

77. A summary of § 74 and § 76 is made in the following statement: *Through a given point, one straight line, and only one, can be drawn perpendicular to a given straight line.*

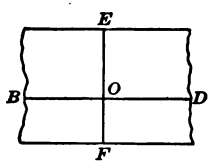
Proof of this statement may be found in § 252 and § 253.

EXERCISES

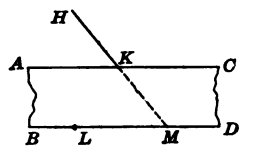
1. Show how the right angle of a carpenter's square may be tested by placing it in the position ABC and drawing a line from A to B. Then turn the square over to the position ABD, so that CB and DB are in the same straight line HK. Is the square correct if the upright edge is in line with AB? Why?



2. Show how to make a line perpendicular to another line by folding a piece of paper forming a straight line BD. Then fold again so that B falls on D. Crease the paper, forming the straight line EF. Why is EF perpendicular to BD?



3. $ACDB$ is a strip of paper, with the edge BD a straight line. It is placed over $\angle HML$ so that BD coincides with LM. The points K and M are marked on the paper. Show how to make with this strip of paper another angle equal to $\angle HML$.



4. If the difference of two supplementary angles is 40°, how many degrees in each angle?

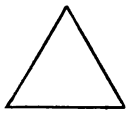
5. If the difference of two complementary angles is 20°, how many degrees in each angle?

6. How many degrees in an angle which is 24° greater than its supplement? In an angle which is 42° less than its supplement?

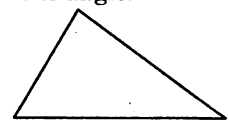
TRIANGLES

78. Triangles Classified According to Relative Length of the Sides. A triangle which has:

(1) all of its sides equal, is an **equilateral** triangle;
(2) two of its sides equal, is an **isosceles** triangle;
(3) no two of its sides equal, is a **scalene** triangle.

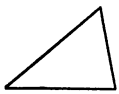

EQUILATERAL

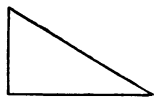
ISOSCELES

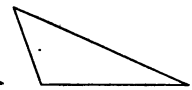

SCALENE

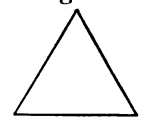
79. Triangles Classified According to Relative Size of the Angles. A triangle which has:

(1) all of its angles acute, is an **acute** triangle;
(2) one right angle, is a **right** triangle;
(3) one obtuse angle, is an **obtuse** triangle;
(4) all of its angles equal, is an **equiangular** triangle.


ACUTE

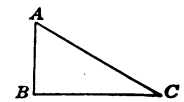

RIGHT


OBTUSE


EQUIANGULAR

80. Hypotenuse. In a right triangle, the side opposite the right angle is the **hypotenuse.**

In the right triangle ABC, AC is the hypotenuse.

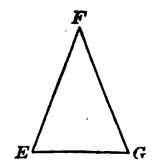


81. The **base** of a triangle is the side on which it is supposed to rest.

In the triangle ABC, BC is the base.

In an isosceles triangle, which has two sides equal, the third side is called the *base*. In all other triangles, however, any side may be considered as the base.

In an isosceles triangle, the angle opposite the base is called the **vertex angle,** and the other two angles are called **base angles.**



In the isosceles triangle EFG, $\angle F$ is the vertex angle; $\angle E$ and $\angle G$ are the base angles.

82. Theorem. A **theorem** is a statement to be proved.

83. Problem. A **problem** is a construction to be made so that it shall satisfy certain given conditions.

Either a theorem or a problem may be called a **proposition.**

84. Problem. *To construct an equilateral triangle when a side is given.*

On the line DF, construct the equilateral triangle DEF. § 25

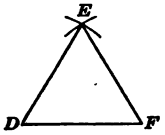


FIG. 1

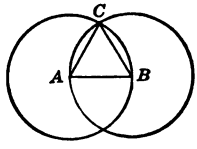


FIG. 2

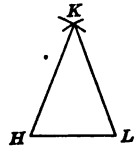
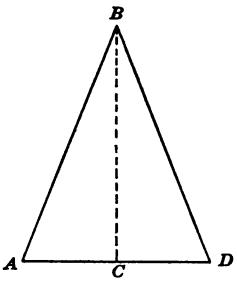


FIG. 3

With the first proposition in his book, Euclid used the method shown in Fig. 2 to construct an equilateral triangle.

85. Problem. *To construct an isosceles triangle when the base and one of the equal sides are given.* (Fig. 3.)

86. Theorem. ***In an isosceles triangle, the angles opposite the equal sides are equal.***



Given the isosceles triangle ABD having $AB = DB$.

To prove $\angle A = \angle D$.

Proof. Draw BC bisecting $\angle ABD$.

Show that $\triangle ABC = \triangle DBC$. § 23

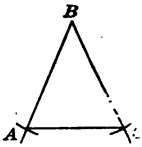
$\therefore \angle A = \angle D$. § 21

87. Theorem. *An equilateral triangle is also equiangular.*

Show that an equilateral triangle is a special form of an isosceles triangle.

88. Theorem. *Each angle of an equilateral triangle is equal to 60°.*

EXERCISES



1. Show how the isosceles triangle ABC is constructed if the vertex angle B is given.

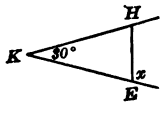
2. Construct an isosceles triangle having the vertex angle a right angle. The vertex angle an obtuse angle.

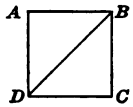
3. Draw an isosceles triangle, using only a straightedge.

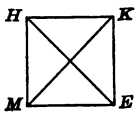
4. Mention some concrete illustrations of the isosceles triangle; *e.g.*, a tripod.

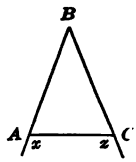
5. If any angle of an isosceles triangle is equal to 60°, the triangle is equiangular.

6. Construct an angle of 60° without using the protractor.









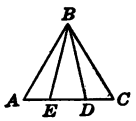
7. In the isosceles triangle HKE, $\angle K = 30°$. How many degrees in $\angle x$.

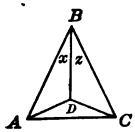
8. Prove that DB divides the square $ABCD$ into two equal isoscele triangles.

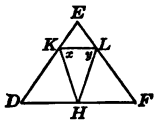
9. In the square $HKEM$, prove that $HE = KM$.

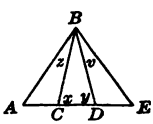
10. Triangle ABC is isosceles. Prove that $\angle x = \angle z$.

11. Triangle ABC is isosceles. $AE = DC$. Prove that $\triangle EBD$ is isosceles





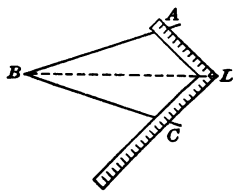




12. The two isosceles triangles ABC and ADC are constructed upon the same base AC. Prove that BD bisects $\angle ABC$.

13. Triangle DEF is isosceles. H is the middle point of DF $DK = FL$. Prove that $\angle x = \angle y$.

14. Triangle ABE is isosceles. The lines BC and BD are drawn, making $\angle z = \angle v$. Prove that $\angle x = \angle y$.

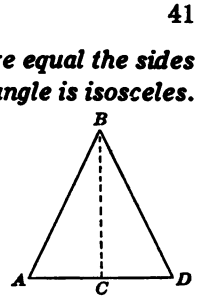


15. Prove that the following method is correct for bisecting an angle with a steel square. Take $BA = BC$. Place the square so that $DA = DC$. Will the line BD bisect $\angle ABC$? Why?

89. Theorem. ***If two angles of a triangle are equal the sides opposite the equal angles are equal and the triangle is isosceles.***

Given $\triangle ABD$ having $\angle A = \angle D$.

To prove that $AB = DB$, and that $\triangle ABD$ is isosceles.



Proof. Draw $BC \perp AD$.

$\angle A = \angle D$. Why?

$\angle BCA = \angle BCD$.

(Being right angles, since BC was constructed $\perp AD$.)

$\angle ABC = \angle DBC$. § 32

(If two angles of one triangle are equal respectively to two angles of another, the third angles are equal.)

$BC = BC$. Iden.

Then $\triangle ABC = \triangle DBC$. § 19

$\therefore AB = DB$. § 21

$\therefore \triangle ABD$ is isosceles. § 78

90. Theorem. *An equiangular triangle is also equilateral.*

EXERCISES

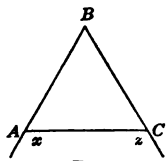


FIG. 1

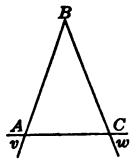


FIG. 2

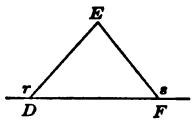


FIG. 3

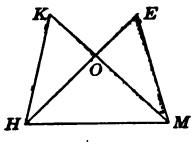
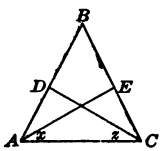


FIG. 4

1. In Fig. 1, $\angle x = \angle z$. Prove that $\triangle ABC$ is isosceles.
2. In Fig. 2, $\angle v = \angle w$. Prove that $\triangle ABC$ is isosceles.
3. In Fig. 3, $\angle r = \angle s$. Prove that $\triangle DEF$ is isosceles.
4. In Fig. 4, $\triangle HOM$ is isosceles. $KO = EO$. Prove that $KH = EM$.
5. In the isosceles triangle ABC, AE and CD are drawn, making $\angle x = \angle z$. Prove that $AE = CD$.



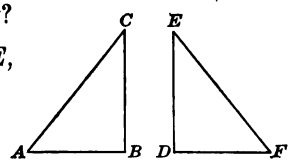
6. Two straight lines drawn from a point in a perpendicular to a given line, cutting off on the given line equal lengths from the foot of the perpendicular, are equal, and make equal angles with the perpendicular.

91. Theorem. *The sum of the acute angles of a right triangle is equal to 90°, or one right angle.* § 30

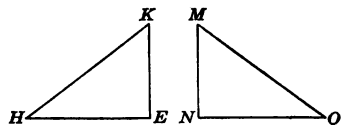
92. Theorem. *If two right triangles have one acute angle of one equal to one acute angle of the other, the other acute angles, also, are equal.* § 32

93. Theorem. *Each acute angle of an isosceles right triangle is equal to 45°.* Why?

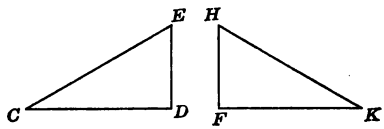
94. Prove that rt.$\triangle ABC$ = rt.$\triangle FDE$, if one side BC and an adjacent acute angle C, are equal respectively to the corresponding side DE, and the corresponding acute angle E. § 19



95. Prove that rt.$\triangle HEK$ = rt.$\triangle ONM$, if one side KE, and the opposite acute angle H, are equal respectively to the corresponding side MN and the opposite acute angle O. (Use § 32 and § 19.)



96. Prove that rt.$\triangle CDE$ = rt.$\triangle KFH$, if the hypotenuse CE and an acute angle C, are equal respectively to the hypotenuse KH and the acute angle K. (Proof similar to § 95.)

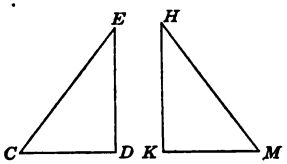


97. From § 94, § 95, and § 96, the following must be true:

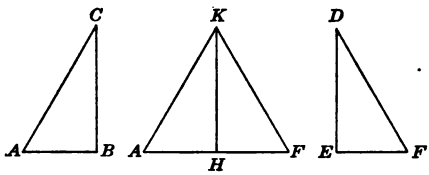
If two right triangles have a side and an acute angle of one equal respectively to the corresponding side and the corresponding acute angle of the other, the triangles are equal.

98. Theorem. *If two triangles have a side and any two angles of one equal respectively to the corresponding side and the two corresponding angles of the other, the triangles are equal.* § 32

99. Prove rt.$\triangle CDE$ = rt.$\triangle MKH$, if the sides CD and DE, adjacent to the rt. $\angle D$, are equal respectively to the sides MK and KH, adjacent to the rt. $\angle K$. § 23



100. Prove that rt.$\triangle ABC$ = rt.$\triangle FED$, if the hypotenuse AC and side BC are equal respectively to the hypotenuse FD and side ED.



Proof. Place the triangles so that the two equal sides, BC and ED, take the position HK, the vertices A and F falling on opposite sides of HK.

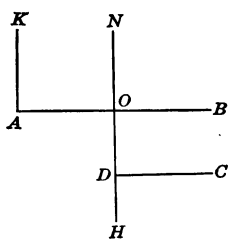
	$\angle KHF$ and $\angle KHA$ are right angles.	Given
Then	AHF is a straight line.	§ 51

(If two adjacent angles are supplementary, their exterior sides are in the same straight line.)

	$\triangle AKF$ is an isosceles triangle.	Def.
Then	$\angle A = \angle F$.	§ 86
	$\triangle AHK = \triangle FHK$.	§ 97
	$\therefore \triangle ABC = \triangle FED$.	

101. From § 99 and § 100, the following must be true: ***If two right triangles have two sides of one equal respectively to two corresponding sides of the other, the triangles are equal.***

102. In drawing lines according to points of the compass, follow the directions of geographical maps. Vertical lines indicate north and south. Horizontal lines indicate east and west.



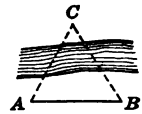
For example, B is east of O. A is west of O. H is south of O. N is north of O. K is north of A. C is east of D.

EXERCISES

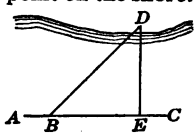
1. C is 4 miles north of A. D is 3 miles east of A, and B is 3 miles west of A. C is 5 miles from B. How far is C from D? Prove your answer.

2. A is 30 miles west of B, and C is 30 miles east of B. D is 20 miles south of C. E is north of A, and on the line DB produced. How far is E from A? Prove your answer.

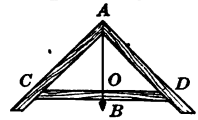
3. Show how to find the distance from A across a stream of water to C by forming the equilateral triangle ABC.



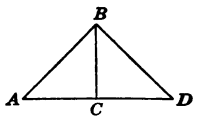
4. A ship was traveling in the direction AC. D is a point on the shore. At B, a sailor observed that the direction BD of the point on shore, made an angle of 45° with the course AC of the ship. At E, $\angle AED$ was observed to be 90°. The ship traveled four miles in going from B to E. Show how the sailor could tell the distance from E to D. (Use § 31 and § 89.)

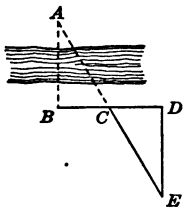


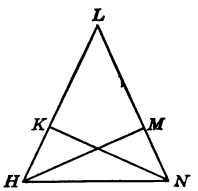
5. This level may be made out of a piece of cardboard or a framework of wood, forming an isoseeles triangle ADC. When the plumb line AB passes through O, the middle point of the base CD, the two triangles AOC and AOD are equal, and $AB \perp CD$. Why?

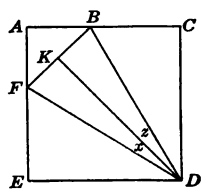


6. In this figure, BC is perpendicular to AD, bisects AD, and is equal to one-half AD. Prove that ABD is an isosceles right triangle.









7. A person wishes to find the distance from B across a stream of water to A. Show by the figure how the distance may be found without crossing the stream. $\angle B$ and $\angle D$ are right angles. C is the middle point of BD.

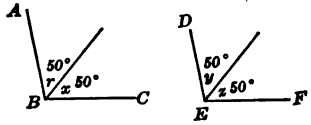
8. Prove that the perpendiculars drawn to the equal sides of an isosceles triangle from the vertices of the opposite angles are equal.

9. In the square $ACDE$, $BC = FE$, and $\angle x = \angle z$. Prove that K is the middle point of FB.

AXIOMS OF EQUALITY

103. Axiom. An **axiom** is a truth which can be established by experiment only. When a truth is so simple that it cannot be derived from truths still simpler, it is an axiom.

104. $\angle x = 50°$.
$\angle z = 50°$.
$\therefore \angle x = \angle z$.



Axiom. *Quantities which are equal to the same quantity or to equal quantities are equal to each other.*

105. $\angle r + \angle x = \angle y + \angle z$.
$\therefore \angle ABC = \angle DEF$.

Axiom. *If equals are added to equals, the sums are equal.*

106. $\angle ABC - \angle x = \angle DEF - \angle z$.

Axiom. *If equals are subtracted from equals, the remainders are equal.*

107. $2\angle r = 2\angle y$.

Axiom. *If equals are multiplied by equals, the products are equal.*

Special Case: *Doubles of equals are equal.*

108. $\dfrac{\angle ABC}{2} = \dfrac{\angle DEF}{2}$.

Axiom. *If equals are divided by equals, the quotients are equal.**

Special Case: *Halves of equals are equal.*

109. $\angle ABC = \angle r + \angle x$.

Axiom. *The whole is equal to the sum of all its parts.*

110. $\angle ABC > \angle r$; $\angle ABC > \angle x$.

Axiom. *The whole is greater than any of its parts.*

111. $\angle DEF = \angle y + \angle z$. $\angle x = \angle y$. Then $\angle DEF = \angle x + \angle z$.

Axiom. *A quantity may be substituted for its equal in any operation.*

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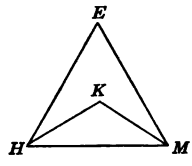
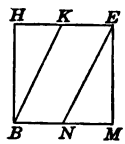
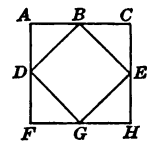
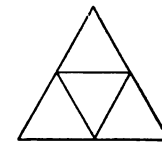
*The divisor must not be zero.

EXERCISES







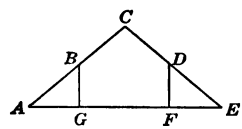


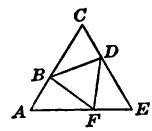
1. The lines KH and KM bisect two of the angles of the equilateral triangle HEM. Find the number of degrees in $\angle K$.

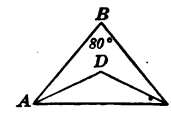
2. The points K and N bisect two sides of the square $HEMB$. Prove that $BK = NE$.

3. The sides of the square $ACHF$ are bisected by the points B, E, G, and D. Prove that $BEGD$ is a square.

4. The lines joining the middle points of the sides of an equilateral triangle divide the triangle into four equal equilateral triangles.



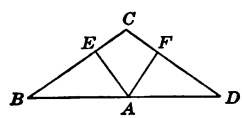


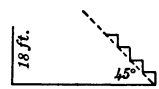


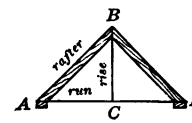
5. Prove that the perpendiculars drawn from the middle points of the equal sides of an isosceles triangle to the base are equal.

6. The points B, D, and F are laid off at equal distances respectively from the vertices A, C, and E of the equilateral triangle ACE. Show that BDF is an equilateral triangle.

7. The lines DA and DC bisect $\measuredangle BAC$ and BCA of the isosceles triangle ABC. $\angle B = 80°$. Find the number of degrees in $\angle D$.





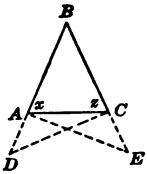


8. Prove that the perpendiculars drawn from the middle point of the base to the equal sides of an isosceles triangle are equal.

9. How many yards of carpet are necessary for the stairway of the figure? Each step is 8 in. high, and the second floor is 18 ft. above the first.

10. The roof shown in the diagram is known as a "half-pitch" roof. The rise is equal to the run. What angle does the rafter make with the horizontal?

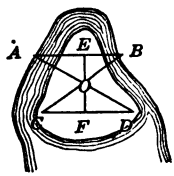
11. In a "half-pitch" roof, what angle do the rafters make at the ridge?



***12.** Prove § 86 by using this figure. The equal sides BA and BC of the isosceles triangle ABC are produced to D and E respectively, making $AD = CE$.

Show that	$\triangle DBC = \triangle EBA$.	§ 105 and § 23
Then	$\triangle DAC = \triangle ECA$.	Why?
And	$\angle DAC = \angle ECA$.	§ 21
	$\therefore \angle x = \angle z$.	§ 55

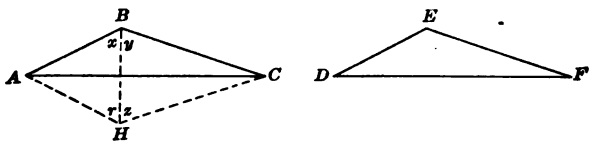
This was the fifth proposition of Euclid's Geometry and came to be known as the *pons asinorum* (bridge of fools).



***13.** Show how a person on an island can find the distance between the two points, A and B, on the mainland.

112. In § 26, the following proposition was shown by experiment to be true. It can now be proved by a process of reasoning.

113. Theorem. ***If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are equal.***



Given $\triangle ABC$ and DEF, having $AB = DE$, $AC = DF$, and $BC = EF$.

To prove $\triangle ABC = \triangle DEF$.

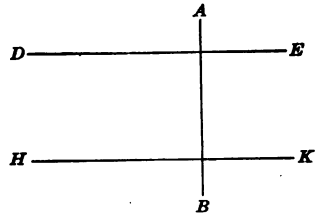
Proof. Place $\triangle DEF$ so that DF coincides with AC and the vertex E falls at H. Draw BH.

	$\triangle ABH$ and CBH are isosceles.	§ 78
	$\angle x = \angle r$, and $\angle y = \angle z$.	§ 86
Hence	$\angle x + \angle y = \angle r + \angle z$.	§ 105
Or	$\angle ABC = \angle AHC$.	Why?
Then	$\triangle ABC = \triangle AHC$.	§ 23
	$\therefore \triangle ABC = \triangle DEF$.	

PARALLEL LINES

114. Plane Surface. A surface such that a straightedge, if placed upon it in any position, touches the surface at all points, is a **plane surface,** or a **plane.**

115. Theorem. *Straight lines in the same plane which are perpendicular to the same straight line can never meet.*



Draw the line AB. With the protractor, draw DE and HK perpendicular to AB.

If DE and HK could meet, there would be two lines perpendicular to AB from the same point. But only one perpendicular can be drawn to a line from a given external point. § 76

$\therefore$ DE and HK cannot meet.

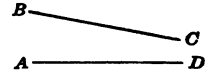
116. Parallel Lines. Straight lines in the same plane which can never meet are **parallel lines.**

Thus DE and HK are parallel lines.

117. Theorem. *Straight lines in the same plane which are perpendicular to the same straight line are parallel.* § 115

118. Axiom. *Through a given point only one straight line can be drawn parallel to another straight line.*

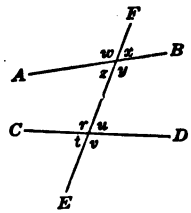
119. Lines such as BC and AD which are not parallel will meet when they are extended if they lie in the same plane.



Axiom. *Straight lines in the same plane which are not parallel will meet if sufficiently prolonged.*

120. Transversal. A straight line which intersects two or more straight lines is a **transversal** of those lines.

121. Angles Formed by a Transversal. The lines AB and CD are cut by the transversal EF. $\measuredangle z$, y, u, and r are **interior** angles. $\measuredangle w$, x, t, and v are **exterior** angles.



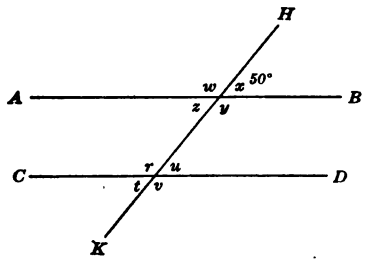
The *alternate interior angles* are z and u, r and y.

The *alternate exterior angles* are w and v, x and t.

The *consecutive interior angles* are z and r, y and u.

The *consecutive exterior angles* are w and t, x and v.

The *exterior interior angles* are w and r, x and u, z and t, y and v.



122. Experiment. Draw two parallel lines, AB and CD. Draw the transversal HK, making $\angle x = 50°$.

How many degrees in each of the $\measuredangle w$, y, and z? With the protractor, measure the number of degrees in $\angle u$. How many degrees in $\measuredangle r$, t, and v? Consider the pairs of angles defined in § 121 and find in this figure the pairs which are equal and the pairs which are supplementary.

If two parallel lines are cut by a transversal:

(1) ***The alternate interior angles are equal.***

(2) ***The alternate exterior angles are equal.***

(3) ***The exterior interior angles are equal.***

(4) ***The consecutive interior angles are supplementary.***

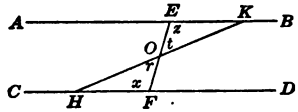
(5) ***The consecutive exterior angles are supplementary.***

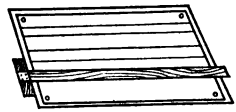
123. Theorem. *If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.* § 122

EXERCISES

1. A triangle cannot have two right angles. Why?

2. The lines AB and CD are parallel. The line EF is bisected at O. Prove that HK, also, is bisected at O.





3. The T-square is used as shown in this figure for drawing parallel lines. Why are the lines parallel?

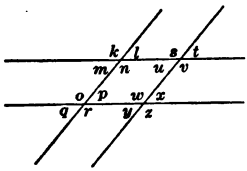
4. Give several illustrations of parallel lines which you have observed.

SUGGESTION. The top and bottom of a page of a book; rails of a railroad track.

5. Give concrete illustrations of parallel lines cut by a transversal.

6. What are some practical illustrations where one pair of parallel lines is intersected by another pair of parallel lines?

7. Which angles are equal in this figure? Name several pairs of supplementary angles.



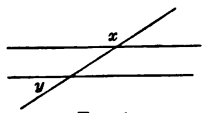


FIG. 1

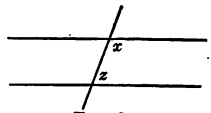
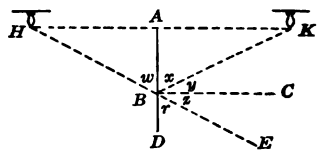


FIG. 2

8. In Fig. 1, $\angle x = 5\angle y$. Find the number of degrees in each angle.

9. In Fig. 2, $\angle x - \angle z = 40°$. Find the number of degrees in each angle.

***10.** It is a law of Physics that if a ray of light is reflected from a mirror, the angle of incidence is equal to the angle of reflection. In the figure, the light from K is reflected by the mirror AD and strikes the eye at E.



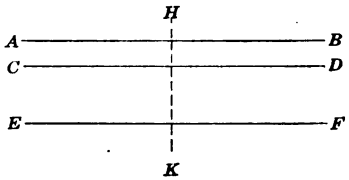
$CB \perp AD$. $\angle y$ is the angle of incidence and $\angle z$ is the angle of reflection. Hence $\angle y = \angle z$.

$\angle x = \angle r$. Why? $\angle w = \angle r$. Why?

$HK \perp AB$. Show that $\triangle HAB = \triangle KAB$.

Why does the electric light appear to be the same distance behind the mirror at H as it really is in front of the mirror at K?

124. Theorem. *Two straight lines parallel to a third straight line are parallel to each other.*



Given AB and CD parallel to EF.

To prove $AB \parallel CD$.

Proof. Draw $HK \perp EF$.

Then HK is perpendicular to AB and CD. § 123

$\therefore AB \parallel CD$. Why?

125. Hypothesis and Conclusion. A theorem is composed of two parts, the *hypothesis* and the *conclusion*. The **hypothesis** is the part that is given. The **conclusion** is the part that is to be proved.

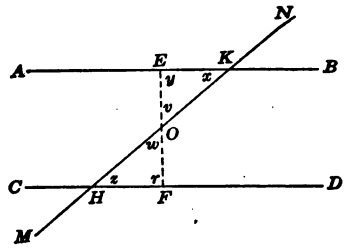
If two parallel lines are cut by a transversal, the alternate interior angles are equal, is a theorem: *if two parallel lines are cut by a transversal,* is the hypothesis; *the alternate interior angles are equal,* is the conclusion.

126. Converse Theorem. *If two straight lines in the same plane are cut by a transversal, making the alternate interior angles equal, the lines are parallel,* is the converse of the theorem in § 125. To form the **converse theorem,** the hypothesis and the conclusion are interchanged.

A theorem is the **converse** of another when the hypothesis and conclusion of one are respectively the conclusion and hypothesis of the other.

It is to be noted that the converse of a theorem is not always true. Thus, *if two angles are right angles, they are equal,* is a theorem which is true. The converse, *if two angles are equal, they are right angles,* is not necessarily true. For this reason, the converse of a theorem cannot be accepted without proof.

127. Theorem. ***If two straight lines in the same plane are cut by a transversal, making the alternate interior angles equal, the lines are parallel.***



Given AB and CD cut by the transversal MN, making $\angle x = \angle z$.

To prove $AB \parallel CD$.

Proof. Through O, the middle point of HK, draw $EF \perp CD$.

$\triangle EOK = \triangle FOH$. § 19

Then $\angle r = \angle y$. Why?

But $\angle r$ is a right angle. Cons.

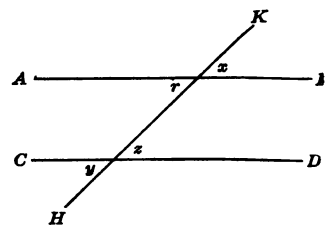
Hence $\angle y$ also is a right angle. § 104

Then AB must be perpendicular to EF. § 38

Since AB and CD are both perpendicular to EF,

$\therefore AB \parallel CD$. § 117

128. Theorem. ***If two straight lines in the same plane are cut by a transversal, making the exterior interior angles equal, the lines are parallel.***



Given AB and CD cut by the transversal HK, making $\angle x = \angle z$.

To prove $AB \parallel CD$.

Proof. $\angle x = \angle r$. Why?

Then $\angle r = \angle z$. Why?

$\therefore AB \parallel CD$. § 127

129. Theorem. ***If two straight lines in the same plane are cut by a transversal, making the alternate exterior angles equal, the lines are parallel.***

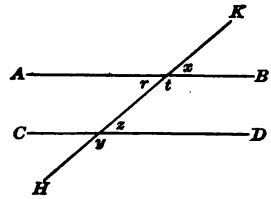
Given $\angle x = \angle y$.

To prove $AB \parallel CD$.

Show that $\angle r = \angle z$. § 104

$\therefore AB \parallel CD$. § 127

130. Theorem. *If two straight lines in the same plane are cut by a transversal, making the consecutive exterior angles supplementary, the lines are parallel.*



Given AB and CD cut by the transversal HK, making angles x and y supplementary.

To prove $AB \parallel CD$.

Proof. Angles y and z are supplementary. Why?

Angles y and x are supplementary. Given

$\angle x = \angle z$. § 55

$\therefore AB \parallel CD$. § 128

131. Theorem. *If two straight lines in the same plane are cut by a transversal, making the consecutive interior angles supplementary, the lines are parallel.*

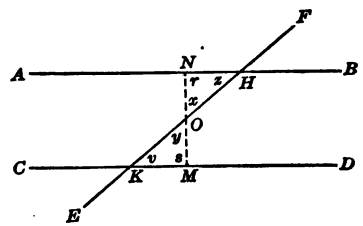
Given $\angle z$ and t supplementary.

To prove $AB \parallel CD$.

Show that $\angle r = \angle z$.

$\therefore AB \parallel CD$. Why?

132. Theorem. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Given the two parallel lines AB and CD cut by the transversal EF in the points H and K.

To prove $\angle v = \angle z$.

Proof. Through O, the middle point of KH, draw $MN \perp CD$.

Then $MN \perp AB$. § 123

Show that rt.$\triangle OMK$ = rt.$\triangle ONH$. § 97

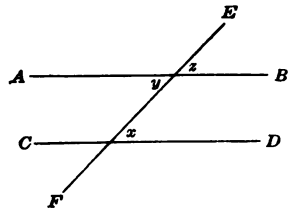
$\therefore \angle v = \angle z$. Why?

133. Theorem. *If two parallel lines are cut by a transversal, the exterior interior angles are equal.*

$\angle x = \angle y$. § 132

$\angle z = \angle y$. Why?

$\therefore \angle x = \angle z$. § 104



134. Theorem. *If two parallel lines are cut by a transversal, the alternate exterior angles are equal.*

Proof similar to § 133.

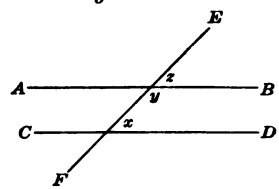
135. Theorem. *If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.*

$\angle x = \angle z$. § 133

Angles z and y are supplementary. Why?

Substitute $\angle x$ for its equal $\angle z$. § 111

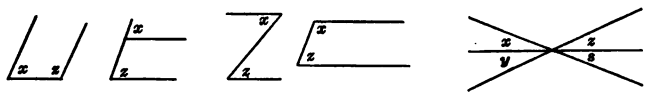
$\therefore$ angles x and y are supplementary.



136. Theorem. *If two parallel lines are cut by a transversal, the consecutive exterior angles are supplementary.*

Proof similar to § 135.

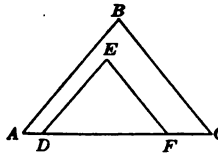
EXERCISES

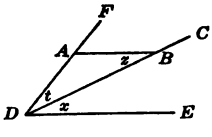


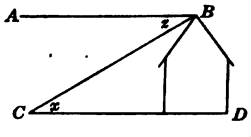
1. The first four figures are special forms of parallel lines cut by a transversal. In each case, tell whether $\measuredangle$ x and z are equal or supplementary, and why.

2. The bisector of one of two vertical angles bisects the other also.

3. Construct an angle of 70° and make the sides two inches long. Construct the bisector of the angle three inches long. Show that two equal triangles are formed if any point on the bisector is connected with the extremities of the sides of the angle.



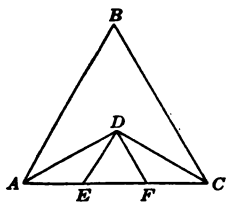


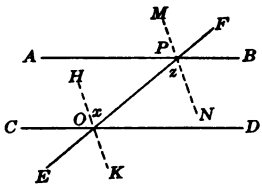


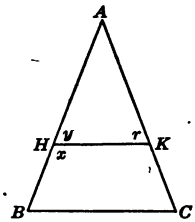
4. Triangle ABC is isosceles, $DE \parallel AB$, $EF \parallel BC$. Prove that $\triangle DEF$ is isosceles.

5. If DC bisects $\angle EDF$ and $AB \parallel DE$, prove that $\triangle ABD$ is isosceles.

6. Lines AB and CD are horizontal, and in the same plane. $\angle x$ is called the *angle of elevation* of point B from point C. $\angle z$ is called the *angle of depression* of point C from point B. Why does $\angle x$ equal $\angle z$?







7. Triangle ABC is equilateral. AD and CD bisect $\angle BAC$ and $\angle BCA$ respectively. $DE \parallel BA$ and $DF \parallel BC$. Prove that $AE = EF = FC$.

8. The parallel lines AB and CD are cut by the transversal EF. The lines HK and MN bisect respectively the alternate interior angles COP and BPO. Prove that $HK \parallel MN$.

SUGGESTION. Show that $\angle x = \angle z$. § 108

9. HK is drawn parallel to BC, the base of the isosceles triangle BAC. Prove that $\triangle HAK$ is isosceles. Also prove $\measuredangle x$ and r supplementary.

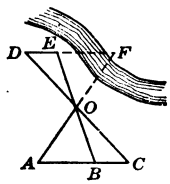
***10.** Show how the distance from E to an inaccessible point F is found by this figure.

The line EB is laid off with O as its middle point.

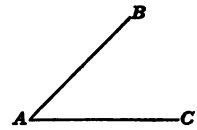
A point D is taken in line with EF and DC is laid off with O as its middle point.

The point A is found in line with both OF and BC.

Prove that $AB = EF$.



137. Right Side and Left Side of Angles. Looking from the vertex of an angle, the side at the right is the **right side** of the angle, and the side at the left is the **left side.**



Thus AC is the *right side* of $\angle A$, and AB is the *left side.*

138. Theorem. *If two angles have their sides parallel, right side to right side, and left side to left side, the angles are equal.*

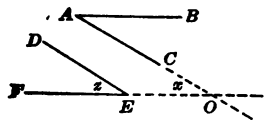


FIG. 1

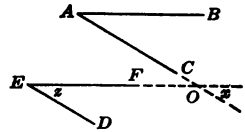


FIG. 2

Given in either figure, $\angle A$ and $\angle z$ having their sides parallel, right side to right side, and left side to left side.

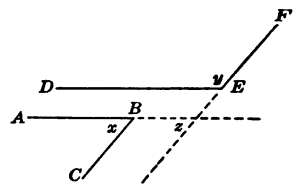
To prove $\angle A = \angle z$.

Proof. Prolong AC and FE until they intersect at some point, as O.

$\angle A = \angle x$.	Fig. 1, § 132, Fig. 2, § 133
$\angle z = \angle x$.	§ 133
$\therefore \angle A = \angle z$.	§ 104

139. Theorem. *If two angles have their sides parallel, right side to left side, and left side to right side, the angles are supplementary.*

Given $\angle x$ and $\angle y$ having their sides parallel, right side to left side, and left side to right side.

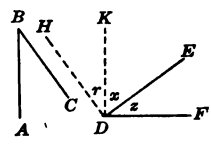


To prove $\measuredangle$ x and y supplementary.

Proof. Prolong AB and FE until they intersect.

$\measuredangle$ z and y are supplementary.	§ 136
$\angle x = \angle z$.	Why?
$\therefore \measuredangle$ x and y are supplementary.	§ 111

140. Theorem. *If two angles have their sides perpendicular, right side to right side, and left side to left side, the angles are equal.*



Given BA, right side of $\angle B$ perpendicular to DF, right side of $\angle z$; and BC, left side of $\angle B$, perpendicular to DE, left side of $\angle z$.

To prove $\angle B = \angle z$.

Proof. Draw $DK \perp DF$, and $DH \perp DE$.

$BA \parallel DK$ and $BC \parallel DH$. § 117

Then $\angle B = \angle r$. § 138

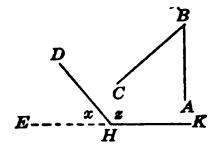
$\measuredangle r$ and x are complementary. Why?

$\measuredangle z$ and x are complementary. Why?

Then $\angle r = \angle z$. § 53

$\therefore \angle B = \angle z$. § 104

141. Theorem. *If two angles have their sides perpendicular, right side to left side and left side to right side, the angles are supplementary.*



Given BA, left side of $\angle B$, perpendicular to HK, right side of $\angle z$; and BC, right side of $\angle B$, perpendicular to HD, left side of $\angle z$.

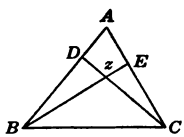
To prove $\measuredangle B$ and z supplementary.

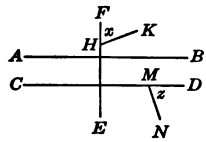
Proof. Prolong KH to E. Then $\angle B = \angle x$. § 140

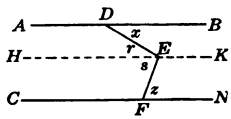
$\measuredangle x$ and z are supplementary. Why?

$\therefore \measuredangle B$ and z are supplementary. § 111

EXERCISES







1. In triangle BAC, $CD \perp AB$, and $BE \perp AC$. Prove that $\measuredangle A$ and z are supplementary.

2. $AB \parallel CD$, $EF \perp AB$, $MN \perp HK$. Prove that $\angle x = \angle z$.

SUGGESTION. Use § 123 and § 140.

3. $AB \parallel CN$. Prove that $\angle DEF = \angle x + \angle z$.

SUGGESTION. Draw $HK \parallel AB$. Why is HK also parallel to CN? § 124

4. Show that the three angles of any triangle in Fig. 1 are equal respectively to the three angles of each of the other triangles. Connect this exercise with § 28.

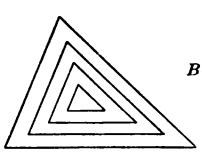


FIG. 1

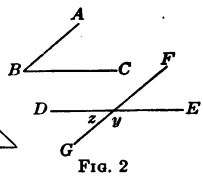


FIG. 2

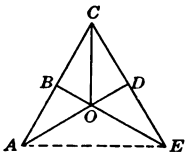


FIG. 3

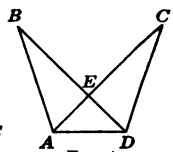


FIG. 4

5. In Fig. 2, $BA \parallel GF$, and $BC \parallel DE$. Prove that $\measuredangle B$ and y are supplementary, and that $\angle B = \angle z$.

6. In Fig. 3, which shows a common form of a "scissors roof truss," $AC = EC$, and $BC = DC$. Prove that $\angle CAD = \angle CEB$, and that $AD = EB$, $AO = EO$, and that OC bisects $\angle ACE$.

7. In Fig. 4, $\angle BAD = \angle CDA$, and $AB = DC$. Prove that $\triangle AED$ is isosceles.

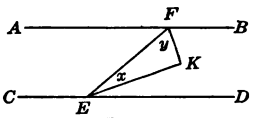


FIG. 5

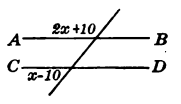


FIG. 6

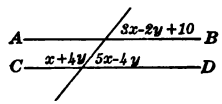


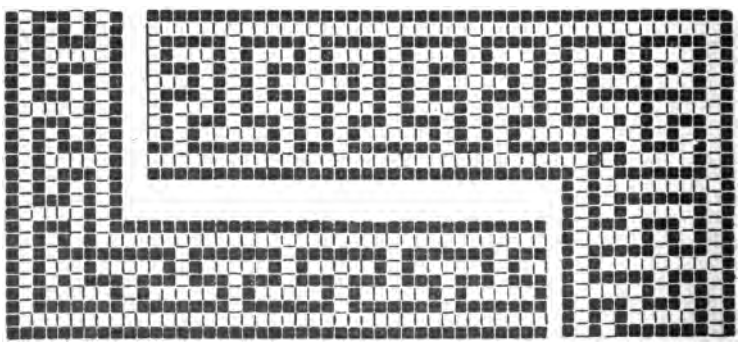
FIG. 7

8. $AB \parallel CD$, FK bisects $\angle EFB$, EK bisects $\angle DEF$. Prove $FK \perp EK$.

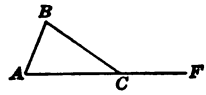
9. If lines AB and CD are parallel, what is the value of x? (Fig. 6.)

10. In Fig. 7, $AB \parallel CD$. At what angle does the transversal intersect the parallel lines?

***11.** Draw the designs of the tile floor borders. They are Greek designs, depending upon perpendicular and parallel lines.



142. Exterior Angles. An angle formed by one side of a figure and an adjacent side produced is an **exterior angle** of the figure.

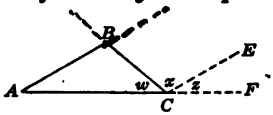


Thus $\angle BCF$ is an exterior angle of $\triangle ABC$.

In $\triangle ABC$, $\angle A$ and $\angle B$ are the **opposite interior angles** in relation to the exterior angle BCF.

143. Theorem. ***The sum of the angles of a triangle is equal to 180°, or two right angles.***

Prolong AC to F. Draw $CE \parallel AB$.



$\angle A = \angle z$. § 133

$\angle B = \angle x$. § 132

$\angle z + \angle x + \angle w = 180°$. § 60

$\therefore \angle A + \angle B + \angle w = 180°$. § 111

144. Theorem. *An exterior angle of a triangle is equal to the sum of the opposite interior angles, and is therefore greater than either of them.*

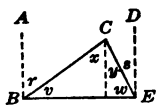
$\angle z + \angle x = \angle A + \angle B$. § 105

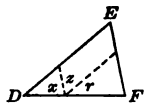
$\therefore \angle BCF = \angle A + \angle B$. Why?

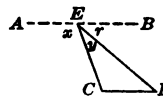
$\therefore \angle BCF > \angle A$ and $\angle BCF > \angle B$. § 110

145. Theorem. *In a triangle, there can be only one right angle, or one obtuse angle.* Why?

EXERCISES







1. Prove the theorem of § 143, using the above figures.

FIG. 1

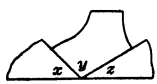
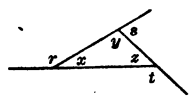
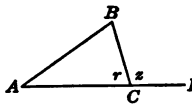


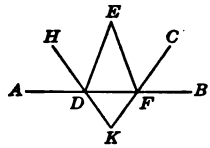
FIG. 2

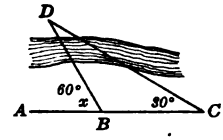


2. Show how the sum of the angles of a triangle may be found to be equal to a straight angle by cutting out a paper triangle, and tearing it into three pieces, as in Fig. 1. Place the three pieces as in Fig. 2.

3. How many degrees in $\angle r + \angle s + \angle t$?



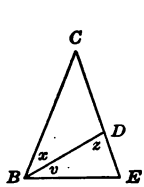


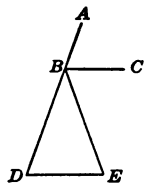


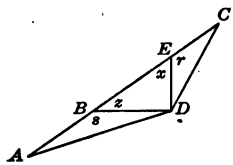
4. $\angle z = 110°$. $\angle B - \angle A = 40°$. How many degrees in each angle of the triangle?

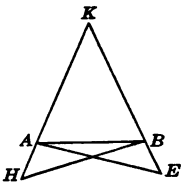
5. Triangle DEF is isosceles. KH and KC bisect the two exterior angles. Prove that $\triangle DKF$ is isosceles.

6. Show how the distance from B to D may be found by the accompanying figure. How many degrees in $\angle D$? Sailors use this method of determining distances, which is known as "doubling the angle on the bow."









7. When each base angle of an isosceles triangle is twice the vertex angle, the bisector of either base angle divides the triangle into two isosceles triangles.

SUGGESTION. Show that $\angle C = 36°$; $\angle x = \angle C$; $\angle z = \angle E$.

8. Triangle DBE is isosceles. BC bisects $\angle ABE$. Prove $BC \parallel DE$.

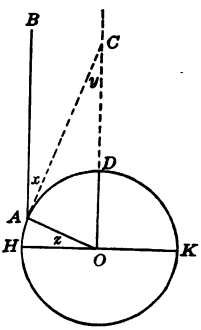
9. Triangle BDE is a right triangle, $BA = BD$, and $EC = ED$. Show that $\angle ADC = 135°$. Also show that $\angle r + \angle s = 270°$.

10. In the isosceles triangle AKB, $AH = BE$. Prove that $\angle ABH = \angle BAE$.

***11.** The latitude of a place may be determined by observing the North Star.

HK is the equator. D is the North Pole. $OD \perp HK$. $AC \perp AO$. AB and OD both point to the North Star and are considered to be parallel on account of the great distance of the star. To the observer at A, $\angle x$ is the angle of elevation of the North Star. Show that $\angle z = \angle y = \angle x$.

When the observer measures $\angle x$ he determines the latitude at A, which is equal to $\angle z$.



QUADRILATERALS

146. Closed Figure. If a portion of a plane surface is entirely separated from the remaining portion by a line or lines, a **closed figure** is formed.

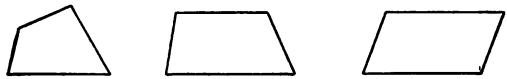
147. Quadrilateral. A closed figure formed by four straight lines is a **quadrilateral.**

148. Quadrilaterals Classified.

A **trapezium** is a quadrilateral having no two sides parallel.

A **trapezoid** is a quadrilateral having two sides and only two sides parallel.

A **parallelogram** is a quadrilateral having its opposite sides parallel.



TRAPEZIUM TRAPEZOID PARALLELOGRAM

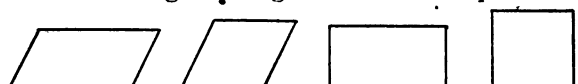
149. Parallelograms Classified.

A **rhomboid** is a parallelogram having no right angles.

A **rhombus** is a rhomboid having all of its sides equal.

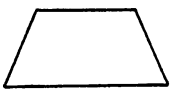
A **rectangle** is a parallelogram having all of its angles right angles.

A **square** is a rectangle having all of its sides equal.



RHOMBOID RHOMBUS RECTANGLE SQUARE

150. An **isosceles trapezoid** is a trapezoid in which the non-parallel sides are equal.

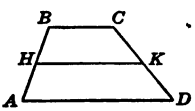


ISOSCELES TRAPEZOID

151. The **median of a trapezoid** is the line joining the middle points of the non-parallel sides.

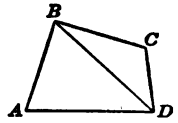
The **bases of a trapezoid** are the parallel sides.

Thus HK is the median of the trapezoid $ABCD$. BC and AD are the bases.



152. The **bases of a parallelogram** are the side on which it is supposed to rest and the opposite side. They are called the **lower base** and **upper base.** Either pair of parallel sides may be considered as the bases.

153. Diagonal. A **diagonal** of a figure is a straight line joining any two vertices not consecutive.



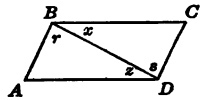
Thus BD is a diagonal of the quadrilateral $ABCD$.

154. Theorem. ***A diagonal divides a parallelogram into two equal triangles.***

Given $\square ABCD$, and a diagonal BD.

To prove $\triangle ABD = \triangle CDB$.

Complete the proof.



155. Theorem. ***In any parallelogram, the opposite sides are equal, and the opposite angles are equal.***

In $\square ABCD$, draw the diagonal BD.

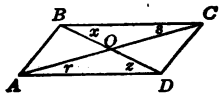
	$\triangle ABD = \triangle CDB$.	§ 154
	$\therefore AB = CD$, $AD = CB$, and $\angle A = \angle C$.	Why?
Also	$\angle r = \angle s$, and $\angle x = \angle z$.	Why?
Then	$\angle r + \angle x = \angle s + \angle z$.	§ 105
	$\therefore \angle ABC = \angle ADC$.	Why?

156. Theorem. *The diagonals of a parallelogram bisect each other.*

Given AC and BD, the diagonals of $\square ABCD$.

To prove $AO = OC$ and $BO = OD$.

Show that $\triangle BOC = \triangle AOD$.

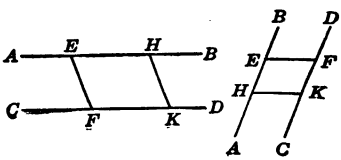


157. Theorem. *Any two consecutive angles of a parallelogram are supplementary.* § 135

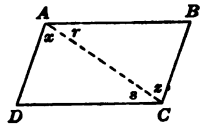
158. Theorem. *Parallels cut off by two parallel lines are equal.*

$AB \parallel CD$ and $EF \parallel HK$.

Prove that $EF = HK$. § 155



159. Theorem. ***If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.***



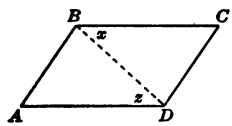
Given $AB = DC$, and $AD = BC$.

To prove that $ABCD$ is a parallelogram.

Proof. Draw the diagonal AC.

	$\triangle ABC = \triangle CDA$.	Why?
Then	$\angle x = \angle z$ and $\angle r = \angle s$.	Why?
	$AD \parallel BC$ and $AB \parallel DC$.	§ 127
	$\therefore ABCD$ is a parallelogram.	§ 148

160. Theorem. ***If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.***



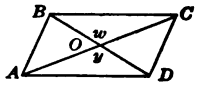
Given the quadrilateral $ABCD$, having $BC = AD$ and $BC \parallel AD$.

To prove that $ABCD$ is a parallelogram.

Proof. Draw the diagonal BD.

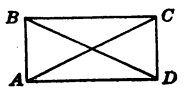
	$\triangle ABD = \triangle CDB$.	Why?
	$AB = CD$.	Why?
	$\therefore ABCD$ is a parallelogram.	§ 159

161. Theorem. ***If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.***



162. Theorem. *The line joining the middle points of two opposite sides of a parallelogram is parallel to and equal to each of the other two sides.*

163. Theorem. *The diagonals of a rectangle are equal.*



164. Theorem. *If a parallelogram has equal diagonals, it is a rectangle.*

165. Theorem. *The diagonals of a square are perpendicular to each other, and bisect the angles of the square.*

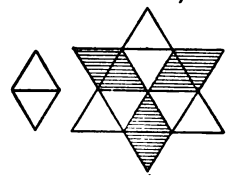
EXERCISES

1. How may a window casing or doorway be tested to see if it is properly "square-cornered?" § 164

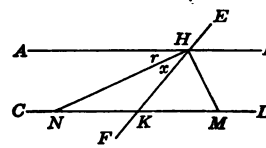
2. The diagonals of a rhombus are perpendicular to each other, and bisect the angles of the rhombus.

3. Two equilateral triangles that have a common side between them form a rhombus.

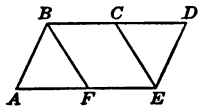
A rhombus of this shape is called a *diamond* and forms the basis of many geometric designs.

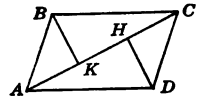


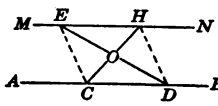
4. The parallel lines AB and CD are cut by the transversal EF. NH bisects $\angle AHK$. MH bisects $\angle BHK$. Prove that $\triangle MHK$ and NHK are isosceles and that $NK = KM$.



5. In parallelogram $ABDE$, C and F are the middle points of BD and AE respectively. Prove that $\triangle ABF = \triangle DEC$; $FB = EC$; and that $FBCE$ is a parallelogram.

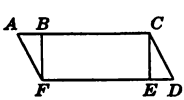


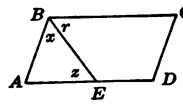


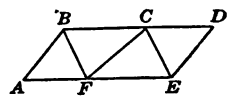


6. The perpendiculars drawn to the diagonals of a parallelogram from opposite vertices are equal.

7. Show that a line may be drawn through a given point parallel to a given line as follows: AB is the given line and H is the given point. Any line, as HC, is drawn from the point H to the line AB. Another point D, on the line AB, is joined to O, the middle point of CH. DO is then produced to E, making OE equal to OD. MN is drawn through E and H. Prove that $MN \parallel AB$.





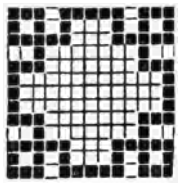


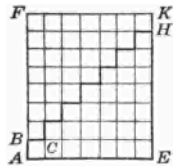
8. In parallelogram $ACDF$, $FB \perp AC$ and $CE \perp FD$. Prove that $\triangle ABF = \triangle DEC$.

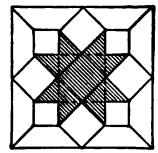
9. In parallelogram $ABCD$, BE bisects $\angle ABC$. Prove that $\triangle ABE$ is isosceles.

10. In parallelogram $ABDE$, BF bisects $\angle ABC$, and CE bisects $\angle DEF$. Prove that $\triangle BFC = \triangle ECF$. Also prove that $BCEF$ is a parallelogram.

11. What part of this tiling is black and what part is white?

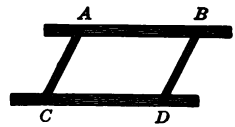






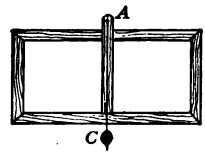
12. In going across a city from A to K, does it make any difference in the distance whether the person goes by the line $ABC \cdots HK$ or from A to E and then to K? Show that $AB+BC \cdots +HK=AE+EK$.

13. In the oak and mahogany parquetry design, show that $\frac{2}{3}$ of the figure is mahogany. (The mahogany is shaded.)

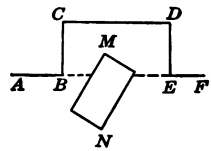


14. The figure $ABDC$ represents a parallel ruler, jointed at A, B, D, and C. Why does AB always remain parallel to CD?

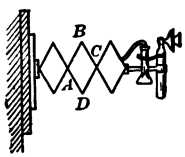
15. Show how the rectangular frame in this figure may be used to level a surface. The plumb bob C is hung from A the middle point of the top of the frame. Is the frame level if the line hangs over the middle point of the bottom of the frame?

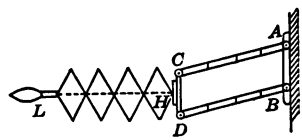


16. Show that the line AB may be produced beyond the building MN as follows: Construct $BC \perp AB$ and then $CD \perp BC$. Construct $DE \perp CD$ and make $DE=BC$. Construct $EF \perp DE$. Prove that AB and EF are in the same straight line.



*17. As the telephone extension opens and closes, why does $ABCD$ always remain a parallelogram? Does it extend in a horizontal line? Why?





*18. The figure shows an electric light bracket, adjustable both horizontally and vertically. Why does LH always remain perpendicular to AB?

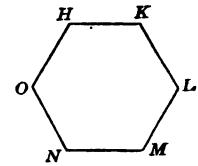
POLYGONS

166. Polygon. A **polygon** is a closed plane figure formed by straight lines. The lines which form the polygon are the **sides** of the polygon. The **perimeter** of a polygon is the sum of its sides.

167. A polygon which has all its sides equal is an **equilateral** polygon.

A polygon which has all its angles equal is an **equiangular** polygon.

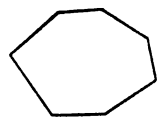
A **regular** polygon is both equilateral and equiangular.



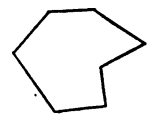
Thus *HKLMNO* is a regular polygon.

168. Polygons Classified. A polygon which has:

Three sides is a **triangle.**
Four sides is a **quadrilateral.**
Five sides is a **pentagon.**
Six sides is a **hexagon.**
Eight sides is an **octagon.**
Ten sides is a **decagon.**
Twelve sides is a **dodecagon.**
Fifteen sides is a **pentadecagon.**



CONVEX POLYGON



CONCAVE POLYGON

169. A **convex polygon** is a polygon none of whose sides when produced will enter the polygon.

A **concave polygon** is a polygon in which two or more of the sides when produced will enter the polygon.

Polygons are considered to be convex unless otherwise stated.

EXERCISES

1. Draw a triangle, a quadrilateral, a pentagon, a hexagon, an octagon, a decagon.

2. Show by a drawing, how many diagonals can be drawn from one vertex in each polygon of Exercise 1. How many diagonals can be drawn from one vertex in a polygon having n sides, *i.e.*, any number of sides?

3. When all the diagonals are drawn from one vertex, how many triangles are formed in each polygon of Exercise 1? In a polygon of n sides, how many triangles are formed.

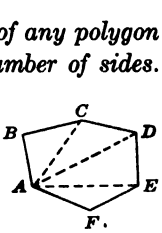
4. Show by a drawing, how many diagonals can be drawn from all the vertices in the polygons of Exercise 1. How many diagonals can be drawn in a polygon of n sides?

5. Tabulate results of Exercises 2, 3, and 4 as follows:

No. of sides	No. of diagonals from one vertex	No. of triangles formed by diag. from one vertex	No. of diagonals from all vertices
3			
4			
5			
6			
8			
10			
n			$\frac{n(n-3)}{2}$

170. Theorem.* *The sum of the interior angles of any polygon is equal to 180° multiplied by two less than the number of sides.*

Given $ABCDEF$, a polygon having n sides.



To prove that the sum of the interior angles is equal to 180° multiplied by $(n-2)$.

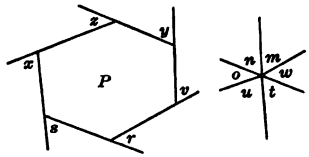
Proof. Draw all the diagonals from any one vertex, as A. This forms $(n-2)$ triangles. What is the sum of the angles of each triangle? The sum of the angles of all the triangles is equal to the sum of the interior angles of the polygon. § 109

$\therefore$ the sum of the interior angles of the polygon is equal to 180° multiplied by $(n-2)$. This may be written $(n-2)$ 180°, or $(n-2)2$ rt.$\measuredangle$.

171. Theorem. *In an equiangular polygon, each angle is equal to* $\frac{(n-2)180°}{n}$, *or* $\frac{(n-2)\,2 \text{ rt.}\measuredangle}{n}$.

*The exercises immediately preceding should be studied before this theorem is assigned.

172. Theorem. *The sum of the exterior angles of a polygon, formed by prolonging one side at each vertex, is 360°.*



Given a polygon P having n sides.

To prove $\angle x+\angle z+\angle y+\angle v+\angle r+\angle s=360°$.

Proof. The figure at the right is formed by lines parallel to the sides of the polygon.

$\angle x=\angle u$; $\angle z=\angle o$; etc. § 138

$\angle u+\angle o+\angle n+\angle m+\angle w+\angle t=360°$. § 59

$\therefore \angle x+\angle z+\angle y+\angle v+\angle r+\angle s=360°$. § 111

EXERCISES

1. Find the sum of the interior angles of a quadrilateral, a pentagon, a hexagon, an octagon, a decagon, a pentadecagon.

2. Find the value of each interior angle and each exterior angle of a regular pentagon, hexagon, octagon, decagon, pentadecagon, and also of a regular polygon of twenty sides. Tabulate results of Exercises 1 and 2.

3. Three angles of a quadrilateral are 110°, 70°, and 110° respectively. What is the size of the fourth angle? If the angles of 110° are opposite each other, what kind of quadrilateral is it? If the angles of 110° are consecutive, what kind of quadrilateral is it?

4. How many sides has the regular polygon each of whose interior angles is 140°? 150°? 90°? 120°?

5. How many sides has the regular polygon each of whose exterior angles is 90°? 120°? 40°? 36°? 24°?

6. The sum of two angles of a triangle is 150° and their difference is 70°. How many degrees has each angle of the triangle?

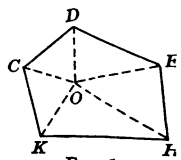


FIG. 1

7. Prove § 170 with the construction given in Fig. 1.

8. In Fig. 2, why does $\angle x=\angle z$? In § 172, does it make any difference which side is prolonged at any vertex?

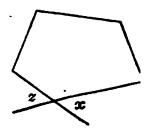


FIG. 2

INEQUALITIES

173.

$$\begin{array}{r} 6< \ 8<10<12<14 \\ +\ \underline{4=\ 4=\ 4=\ 4=\ 4} \\ 10<12<14<16<18 \end{array}$$

Axiom. *If equals are added to unequals, the sums are unequal in the same order.*

The expression, "in the same order," signifies that the unequals in the result are increasing if the unequals which are given are increasing, and that the unequals in the result are decreasing if the unequals which are given are decreasing. Reading from left to right in the illustration, the unequals are increasing; and from right to left, they are decreasing.

174.

$$\begin{array}{r} 6< \ 8<10<12<14 \\ -\ \underline{4=\ 4=\ 4=\ 4=\ 4} \\ 2<\ 4<\ 6<\ 8<10 \end{array}$$

Axiom. *If equals are subtracted from unequals, the remainders are unequal in the same order.*

175.

$$\begin{array}{r} 6< \ 8<10<12<14 \\ \times\ \underline{4=\ 4=\ 4=\ 4=\ 4} \\ 24<32<40<48<56 \end{array}$$

Axiom. *If unequals are multiplied by positive equals, the products are unequal in the same order.*

176. $2\overline{)6}$ $\quad 3$ $\qquad 2\overline{)8}$ $\quad 4$ $\qquad 2\overline{)10}$ $\quad 5$ $\qquad 2\overline{)12}$ $\quad 6$ $\qquad 2\overline{)14}$ $\quad 7$

Axiom. *If unequals are divided by positive equals, the quotients are unequal in the same order.*

177.

$$\begin{array}{r} 6< \ 8<10<12<14 \\ +\ \underline{3<\ 4<\ 5<\ 6<\ 7} \\ 9<12<15<18<21 \end{array}$$

Axiom. *If unequals are added to unequals in the same order, the sums are unequal in that order.*

178.
$$\begin{aligned} 20 &= 20 = 20 = 20 = 20 \\ -\ 6 &< 8 < 10 < 12 < 14 \\ 14 &> 12 > 10 > 8 > 6 \end{aligned}$$

Axiom. *If unequals are subtracted from equals, the remainders are unequal in the reverse order.*

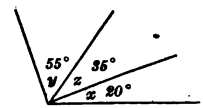
The expression, "in the reverse order," signifies that the unequals in the result are increasing if the unequals which are given are decreasing, and that the unequals in the result are decreasing if the unequals which are given are increasing.

179. $\angle y = 55°$, $\angle z = 35°$, $\angle x = 20°$.

$\angle y > \angle z$.

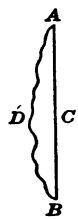
$\angle z > \angle x$.

$\therefore \angle y > \angle x$.



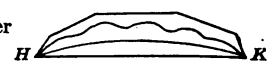
Axiom. *If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.*

180. Straight Line. The expression **straight line** is derived from words meaning a *stretched linen cord*, or simply *stretched linen*. ACB is a stretched string, and ADB is a loose string joining the points A and B. If lines are marked underneath the strings, they must have the same lengths as the strings. Upon stretching the loose string ADB and comparing its length with ACB, it is found that the straight line ACB is shorter than the line ADB.



181. Axiom. *A straight line is the shortest line that can be drawn from one point to another.*

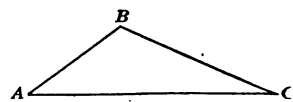
The straight line drawn from H to K is shorter than the other lines drawn from H to K.



182. Theorem. *The sum of two sides of a triangle is greater than the third side. The difference between two sides of a triangle is less than the third side.*

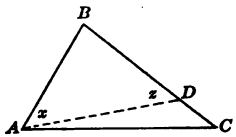
$AC < AB + BC$ § 181

$AC - AB < BC$; $AC - BC < AB$. § 174



183. Theorem. *If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the greater angle is opposite the greater side.*

Given $\triangle ABC$, having the side BC greater than the side AB.



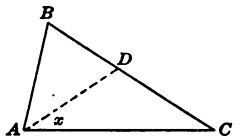
To prove $\angle A > \angle C$.

Proof. On BC, lay off $BD = BA$ and draw AD.

	$\triangle ABD$ is isosceles.	Why?
Then	$\angle x = \angle z$.	Why?
	$\angle z > \angle C$.	§ 144
Then	$\angle x > \angle C$.	§ 111
But	$\angle A > \angle x$.	§ 110
	$\therefore \angle A > \angle C$.	§ 179

184. Theorem. *If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the greater side is opposite the greater angle.*

Given $\triangle ABC$, having $\angle A > \angle C$.



To prove $BC > BA$

Proof. Draw AD, making $\angle x = \angle C$.

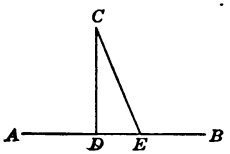
$\triangle ADC$ is isosceles. Why?

Then	$AD = DC$.	Why?
	$BD + DA > BA$.	Why?
	$BD + DC > BA$.	§ 111
	$\therefore BC > BA$.	

185. Theorem. *A perpendicular is the shortest line that can be drawn from a point to a straight line.*

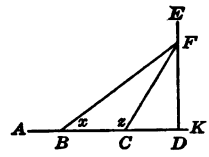
Given CD drawn perpendicular to the line AB from the point C, and CE any other line drawn from the point C to the line AB.

To prove $CD < CE$.



Proof.	$\angle CDE$ is a right angle.	Given
	$\angle CED$ is an acute angle.	§ 145
Then	$\angle CED < \angle CDE$.	Why?
	$\therefore CD < CE$.	§ 184

186. Theorem. *If two straight lines are drawn from a point in a perpendicular to a given line, cutting off on the line unequal lengths from the foot of the perpendicular, the more remote is the greater.*

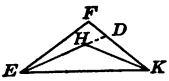


$\angle x$ is acute. Why? $\angle z$ is obtuse. Why?

$\therefore BF > CF.$ § 184

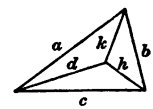
EXERCISES

1. Can the sides of a triangle be respectively: 6 in., 8 in., 10 in.? 7 in., 3 in., 12 in.? 4 in., 11 in., 3 in.? 5 in., 4 in., 9 in.? 2 in., 10 in., 15 in.? 3 in., 9 in., 7 in.?

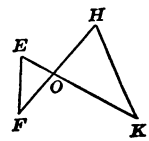


2. The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn but included by them.

Prove that $EF+FK>EH+HK$.



3. The sum of the lines drawn from a point within a triangle to the three vertices is greater than half the perimeter.



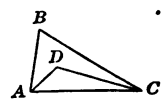
4. Each side of a triangle is less than half the perimeter.

5. Show that $EK+HF>EF+HK$.

6. If an isosceles triangle has a right angle or an obtuse angle, each of the equal sides is less than the third side.

7. If each of the equal sides of an isosceles triangle is greater than the third side, the vertex angle is less than 60°.

8. If the diagonals of a parallelogram are unequal, the figure is a rhomboid.



9. In the figure, $BC>BA$. DA bisects $\angle A$. DC bisects $\angle C$. Prove that $DC>DA$.

10. The straight line joining the vertex of an isosceles triangle to any point in the base is less than either of the equal sides of the triangle.

DISTANCE

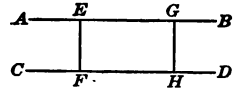
187. Distance. Distance is measured on the shortest line that can be drawn between two points, two lines, or a point and a line.

188. The **distance between two points** is the length of the straight line joining them. § 181

189. The **distance from a point to a line** is the length of the perpendicular from the point to the line. § 185

190. The **distance between two parallel lines** is the length of the perpendicular to either one of them from any point on the other. § 185

191. Theorem. *Two parallel lines are everywhere equally distant.*



Given AB and CD two parallel lines.

To prove that they are everywhere equally distant.

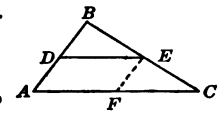
Proof. Draw EF and GH perpendicular to CD from any points, as E and G, on the line AB.

Then $EF \parallel GH$. Why? $\therefore EF = GH$. § 158

192. Theorem. ***If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.***

Given DE bisecting BC and parallel to AC.

To prove that DE bisects AB.



Proof. Draw $EF \parallel AB$.

$\triangle BDE = \triangle EFC$. Why?

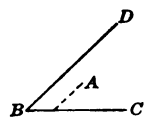
Then $DB = FE$. Why?

But $AD = FE$. Why?

$\therefore DB = AD$, and DE bisects AB. Why?

EXERCISE

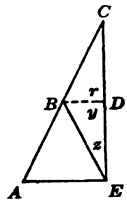
Through any point within an angle, draw a line such that the part of the line lying between the sides of the angle shall be bisected at that point.



193. Theorem. *The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.*

Given B the middle point of AC, the hypotenuse of rt.$\triangle ACE$.

To prove $AB=BE=BC$.



Proof. Draw $BD \parallel AE$. Then $CD=DE$. § 192

$\angle r$ and $\angle y$ are right angles. Why?

rt. $\triangle BDE$ = rt. $\triangle BDC$. § 101

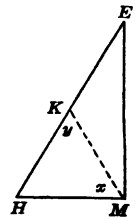
Then $BE=BC$. Why?

$\therefore$ B is equidistant from A, C, and E.

194. Theorem. ***If one acute angle of a right triangle is twice the other, the hypotenuse is twice the shorter side.***

Given rt. $\triangle HEM$, having $\angle H=2\angle E$.

To prove $HE=2HM$.



Proof. Draw MK from the vertex of the right angle to the middle point of the hypotenuse.

$\angle H=60°$ and $\angle E=30°$. Why?

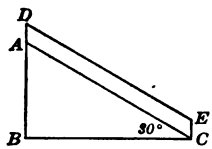
$KH=KM$. § 193

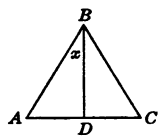
How many degrees in $\angle x$?

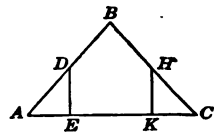
$\triangle HKM$ is equilateral. Why?

$\therefore$ $HK=HM$ and $HE=2HM$.

EXERCISES





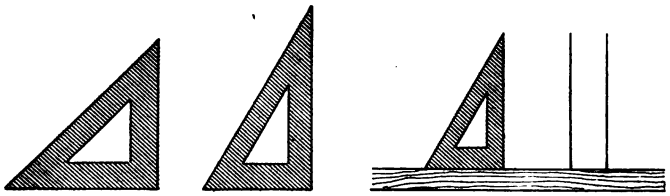


1. A stairway rising at an angle of 30° leads from one floor to another, 12 ft. above. AB is 12 ft. Find the length of the railing, DE.

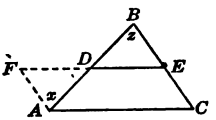
2. Prove § 194 by using the equilateral triangle ABC, in which $BD \perp AC$. Show that $\angle A=60°$ and $\angle x=30°$. Also show that $AB=2AD$.

3. In the isosceles triangle ABC, the equal sides AB and BC are bisected at D and H, respectively. $DE \perp AC$ and $HK \perp AC$. Prove that $EK=AE+KC$.

195. In mechanical drawing, use is made of right triangles made of wood or celluloid. The most commonly used are the isosceles right triangle and the triangle having the acute angles 30° and 60°. What angles can be made by using them separately and in combination? Show how they may be used in drawing parallel lines and perpendicular lines.



196. Theorem. ***The line joining the middle points of two sides of a triangle is parallel to the third side and is equal to one-half the third side.***



Given the line DE joining the middle points of AB and BC.

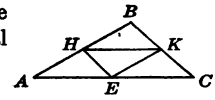
To prove $DE \parallel AC$ and $DE = \frac{1}{2}AC$.

Proof. Prolong ED to F, making $DF = DE$. and draw AF.

	$\triangle BDE = \triangle ADF$.	Why?
Then	$AF = BE$, and $\angle x = \angle z$.	Why?
But	$EC = BE$.	Given
Then	$AF = EC$.	§ 104
Also	$AF \parallel BC$.	§ 127
Then	$AFEC$ is a parallelogram.	§ 160
	$\therefore DE \parallel AC$ and $FE = AC$.	Why?
But	$DE = \frac{1}{2}FE$.	Cons.
	$\therefore DE = \frac{1}{2}AC$.	Why?

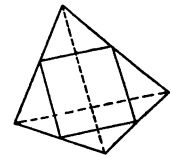
EXERCISES

1. The lines joining the middle points of the three sides of a triangle divide the triangle into four equal triangles.



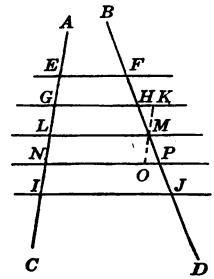
SUGGESTION. Show that $AHKE$ is a $\square$.

2. Construct a triangle when the middle points of the three sides are given.



3. The lines joining the middle points of the sides of any quadrilateral, taken in order, form a parallelogram. § 196

197. Theorem. ***If three or more parallel lines intercept equal parts on one transversal, they intercept equal parts on every transversal.***



Given EF, GH, LM, NP, and IJ, parallel lines cutting the transversals AC and BD, making EG, GL, LN, and NI equal.

To prove that FH, HM, MP, and PJ are equal.

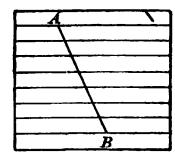
Proof. Draw KO through M parallel to AC.

	$GKML$ and $LMON$ are $\boxed{S}$.	Why?
Then	$KM = GL$, and $MO = LN$.	§ 155
Since it was given that $GL = LN$, then	$KM = MO$.	§ 104
	$\triangle HKM = \triangle POM$.	§ 19
	$\therefore HM = MP$.	Why?

In like manner it can be proved that $FH = HM$, and $MP = PJ$.

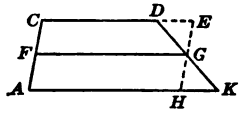
EXERCISE

Show how a line may be divided into equal parts by means of ruled paper. Why is AB divided into seven equal parts?



198. Theorem. *The median of a trapezoid is parallel to the the bases and is equal to half the sum of the bases.*

Given FG the median of the trapezoid $ACDK$.



To prove that FG is parallel to CD and AK, and that $FG=\frac{1}{2}(CD+AK)$.

Proof. Draw $HG \parallel AC$ to meet CD produced, at E.

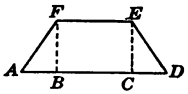
$CEHA$ is a parallelogram.	Def.
$\triangle DGE=\triangle KGH$.	Why?
$\therefore DE=HK$.	Why?
$CD+AK=CE+AH$.	Why?
FG is equal and parallel to CE and AH.	§ 162

$\therefore FG=\frac{1}{2}(CD+AK)$ and FG is parallel to CD and AK.

EXERCISES

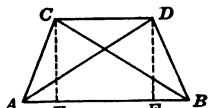
1. In an isosceles trapezoid, the angles at the extremities of either base are equal.

2. If the angles at the extremities of either base of a trapezoid are equal, the trapezoid is isosceles.

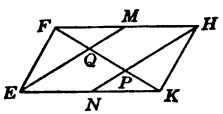


3. The diagonals of an isosceles trapezoid are equal.

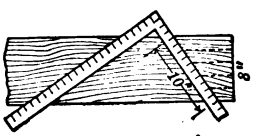
4. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.



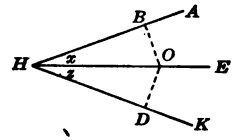
5. In the parallelogram $EFHK$, M and N are the middle points of two opposite sides. Prove that EM and HN trisect the diagonal FK.



6. The carpenter makes use of the construction shown in this figure to divide a board of any width into any number of equal strips. Here an 8-in. board is divided into five equal strips. Why are the strips of equal width? Apply the method to a board 13 in. wide.



199. Theorem. ***All points in the bisector of an angle are equidistant from the sides of the angle.***



Given HE the bisector of $\angle AHK$.

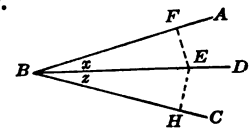
To prove that all points in HE are equidistant from HA and HK.

Proof. From any point, as O, in HE, draw $OB \perp HA$ and $OD \perp HK$. Show that $\triangle BOH = \triangle DOH$. § 97

$\therefore OB = OD$. Why?

Since O, any point in HE, is equidistant from HA and HK, all points in the bisector of $\angle AHK$ are equidistant from the sides of the angle.

200. Theorem. ***All points equidistant from the sides of an angle are in the bisector of the angle.***



Given E any point equidistant from BA and BC, the sides of $\angle ABC$.

To prove that E is in the bisector of $\angle ABC$.

Proof. Draw BD through E from the vertex B. Draw $EF \perp BA$ and $EH \perp BC$.

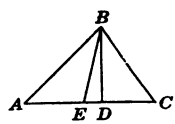
$\triangle EBF = \triangle EBH$. § 101

Then $\angle x = \angle z$. Why?

$\therefore BD$ is the bisector of $\angle ABC$. Why?

Since E, any point equidistant from BA and BC, lies in BD, all points equidistant from BA and BC lie in BD.

201. Altitude of a Triangle. A line drawn from any vertex of a triangle perpendicular to the opposite side is an **altitude of the triangle.** A triangle has three altitudes.



Thus BD is an altitude of triangle ABC.

202. Median of a Triangle. A line drawn from any vertex of a triangle to the middle point of the opposite side is a **median of the triangle.**

Thus BE is a median of the triangle ABC.

203. The **altitude of a parallelogram or a trapezoid** is the distance between the bases.

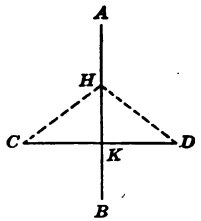
In each figure, a is the altitude.

204. Perpendicular Bisector. A line which bisects another line and is perpendicular to it, is a **perpendicular bisector** of the line.

205. Theorem. *All points in the perpendicular bisector of a straight line are equidistant from the extremities of the line.*

Given AB the perpendicular bisector of CD.

To prove that H, any point in AB, is equidistant from C and D.



Proof. Draw CH and DH.

$\triangle CKH = \triangle DKH$. § 101

Then $CH = DH$. Why?

$\therefore$ H, any point in AB, is equidistant from C and D.

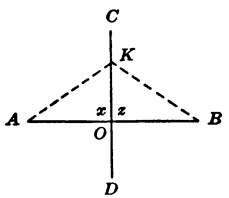
Then all points in AB are equidistant from the extremities of CD.

206. Theorem. *All points equidistant from the extremities of a straight line lie in the perpendicular bisector of the line.*

Given K, any point equidistant from A and B.

To prove K is in the perpendicular bisector of AB.

Proof. Draw CD through K and O, the middle point of AB. Draw AK and BK.



Show that $\triangle AKO = \triangle BKO$. § 27

$\angle x = \angle z$. Why?

Then $CD \perp AB$. Why?

$\therefore$ all points equidistant from A and B lie in CD, the perpendicular bisector of AB.

207. Theorem. ***Two points, each equidistant from the extremities of a straight line, determine the perpendicular bisector of the line.*** § 66

EXERCISES

Illustrate the following by drawing, without proof:

1. Where are all the points equidistant from two parallel lines which are one inch apart? Where are all the points equidistant from any two given parallel lines?

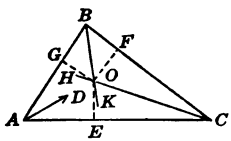
2. Where are all the points one inch distant from a given line? Where are all the points at a given distance from a given line?

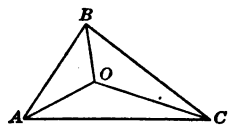
3. Where are all the points equidistant from two points which are one inch apart? Where are all the points equidistant from any two given points?

CONCURRENT LINES

208. Concurrent Lines. If two or more lines pass through the same point, they are **concurrent lines.**

209. Theorem. *The bisectors of the angles of a triangle pass through a common point, which is equidistant from the sides of the triangle.*





Given AD, BK, and CH, the bisectors of the angles of $\triangle ABC$.

To prove that AD, BK, and CH pass through a common point, which is equidistant from AB, BC, and AC.

Proof. BK and CH pass through some common point, as O, since they are not parallel. § 119

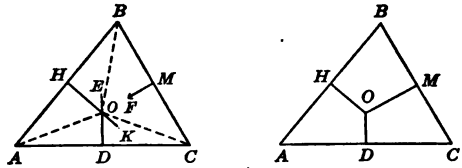
Since O is in the bisector BK, it is equidistant from AB and BC, and hence $OG=OF$. § 199

Since O is in CH, it is equidistant from AC and BC, and hence $OE=OF$. Then $OG=OE$ (Why?), and O is equidistant from AB and AC. Since O is equidistant from AB and AC, it must be in the bisector of $\angle BAC$. § 200

Then AD, the bisector of $\angle BAC$, must pass through O.

$\therefore$ the three bisectors pass through a common point.

210. Theorem. ***The perpendicular bisectors of the three sides of a triangle pass through a common point, which is equidistant from the three vertices of the triangle.***



Given DE, HK, and MF, the perpendicular bisectors of the sides of $\triangle ABC$.

To prove that DE, HK, and MF pass through a common point which is equidistant from A, B, and C.

Proof. DE and HK pass through some common point, as O, since they are not parallel. § 119

Since O is in DE, the perpendicular bisector of AC, it is equidistant from A and C. Then $OC = OA$. § 205

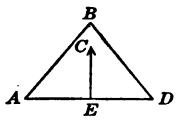
Likewise, O is equidistant from A and B. Then $OB = OA$.

Hence $OB = OC$ (Why?), and O is equidistant from B and C.

Then O must be in the perpendicular bisector of BC. § 206

$\therefore$ MF must pass through O, and the three perpendicular bisectors pass through a common point.

EXERCISES



1. The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the vertex angle. § 206

2. The line joining the vertex of an isosceles triangle to the middle point of the base is perpendicular to the base and bisects the vertex angle.

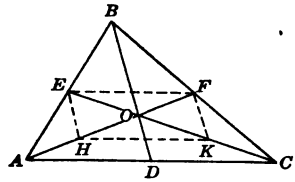
3. The bisector of the vertex angle of an isosceles triangle bisects the base and is perpendicular to it.

4. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.

5. Any point in the bisector of the vertex angle of an isosceles triangle is equidistant from the extremities of the base.

6. If two isosceles triangles have the same base, the line passing through their vertices is the perpendicular bisector of the base.

211. Theorem. *The medians of a triangle pass through a common point, which is two-thirds of the length of each median from its vertex.*



Given BD, AF, and CE, the medians of $\triangle ABC$.

To prove that BD, AF, and CE pass through a common point O, and that $BO=\frac{2}{3}BD$, $CO=\frac{2}{3}CE$, and $AO=\frac{2}{3}AF$.

Proof. Since AF and CE are not parallel, they will pass through some common point, as O. § 119

Join H and K, the middle points of AO and CO respectively. Then draw HE, EF, and FK.

In $\triangle ABC$, EF joins the middle points of AB and BC.

Then $EF \parallel AC$ and $EF=\frac{1}{2}AC$. § 196

In $\triangle AOC$, $HK \parallel AC$ and $HK=\frac{1}{2}AC$.

Then $EF=HK$ and $EF \parallel HK$. Why?

$\therefore EFKH$ is a parallelogram. § 160

Then $HO=OF$ and $EO=OK$. § 156

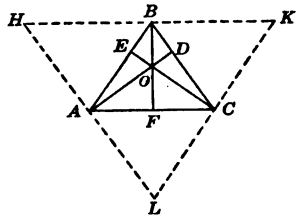
$AH=HO=OF$ and $EO=OK=KC$. Why?

$\therefore AO=\frac{2}{3}AF$ and $CO=\frac{2}{3}CE$.

In like manner, it can be shown that BD passes through O and $BO=\frac{2}{3}BD$.

212. The point of intersection of the three medians of a triangle is the **center of weight** of the triangle. This can be shown as follows: Cut a triangle out of cardboard or tin. Draw the three medians. Place the point of intersection on the point of a pencil and see if the triangle can be made to balance.

213. Theorem. *The three altitudes of a triangle pass through a common point.*



Given AD, BF, and CE, the three altitudes of $\triangle ABC$.

To prove that AD, BF, and CE pass through a common point O.

Proof. Through the three vertices of the triangle, draw HK, KL, and LH parallel to AC, AB, and BC, respectively, forming $\triangle HKL$.

$BF \perp HK$, $AD \perp LH$, $CE \perp KL$. § 123

$AHBC$ and $ABKC$ are parallelograms. § 148

$\therefore HB = AC$, and $BK = AC$. § 155

Then $HB = BK$ and B is the middle point of HK. Why?

In like manner, it can be shown that A and C are the middle points of LH and KL respectively.

Then AD, BF, and CE are the perpendicular bisectors of the sides of $\triangle HKL$.

$\therefore AD$, BF, and CE pass through a common point. § 210

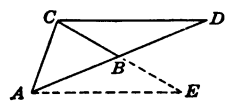
EXERCISES

1. Construct an acute triangle, a right triangle, and an obtuse triangle. Draw the three altitudes in each. Observe the point of meeting.

2. Construct an acute triangle, a right triangle, and an obtuse triangle. Draw the three medians in each.

3. The sum of two sides of a triangle is greater than twice the median drawn to the third side.

Prove that $AC + CD > 2CB$.



4. The perimeter of a triangle is greater than the sum of the three medians.

DEFINITIONS

214. Definition. A **definition** of an object is a description given in such a way that the object is distinguished from all other objects.

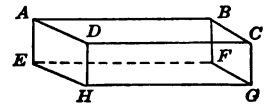
The converse of a correct definition is always true.

For example, an acute angle is an angle which is less than a right angle. And conversely, an angle which is less than a right angle is an acute angle.

215. Geometric Solid. A block of wood or a brick is a solid. According to the general use of the term, a solid is a limited portion of space filled with matter. However, in geometry, no consideration is given to matter. Only the space occupied is studied. Should the brick be placed in plaster and then removed, the opening left in the plaster would be an illustration of a geometric solid.

A **geometric solid** is a limited portion of space.

216. Dimensions. A solid has three dimensions:



(1) **Length:** from D to C, left to right.

(2) **Breadth:** from D to A, front to back; also called **width.**

(3) **Thickness:** from D to H, top to bottom; also called **height, depth,** or **altitude.**

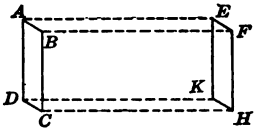
217. Surface. The solid AG has six faces, each of which is a **surface.** A surface such that a straight line joining any two of its points, lies wholly in the surface, is called a **plane surface** or simply a **plane.** The boundaries of a solid are **surfaces.** A surface has two dimensions, *length* and *breadth.*

218. Line. The boundaries of a surface are **lines.** A line has only one dimension, *length.* The intersection of two surfaces is a **line.**

219. Point. The end of a line is a **point.** The intersection of two lines is a **point.** A point has no dimension and shows position only.

220. Geometric Figures. Points, lines, surfaces, and solids, or any combinations of them are **geometric figures.**

221. Generation of Geometric Figures. Let the surface *ABCD* move along the dotted lines to *EFHK*. The surface *ABCD* generates the solid *CE*. The line *BC* generates the surface *BFHC*. The point *B* generates the line *BF*. From the illustration, it can be seen that:



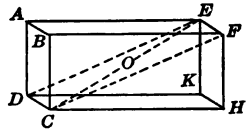
A line may be generated by a moving point.

A surface may be generated by a moving line.

A solid may be generated by a moving surface.

It can also be seen that if a straight line moves in the direction of its own length, it will not generate a surface; also that if a plane surface moves in its own plane, it will not generate a solid.

222. Let the figure *AH* represent a room. Conceive the room as containing absolutely nothing. Then *AH* is a geometric solid. Conceive an imaginary plane passing through *EF* and *DC*. *EFCD* is a geometric surface since it has length and breadth but no thickness. Conceive the line extending from the lower corner *C* to the diagonally opposite upper corner *E*. *CE* is a geometric line, since it has length but neither width nor thickness. Think of *O*, the middle point of the line *CE*. *O* is a geometric point since it has position only, without length, breadth, or thickness.

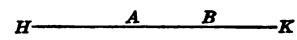


223. Geometry. The science which treats of geometric figures is **Geometry.**

That part of Geometry which treats of figures which lie wholly in the same plane is **Plane Geometry.**

That part of Geometry which treats of figures which do not lie wholly in the same plane is **Solid Geometry.**

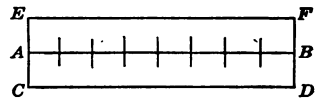
224. Straight Line. A line is a **straight line** if any part, however placed, with its ends on any other part, lies wholly in the line.



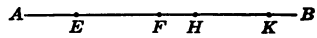
Thus HK is a straight line if any part, as AB, will lie wholly in the line HK, however it may be placed with its ends on HK. The line HK may be tested in this way by using tracing paper.

A ray of light is a straight line. A *line of sight* is a straight line. The line of sight from the eye to a point of light in the distance, is a straight line. In looking through a telescope with cross lines, the line of sight from the point of intersection of the cross lines in the instrument to the point of intersection of any other cross lines is a geometric straight line.

225. Test for a Straightedge. Since a straight line is drawn with a ruler or straightedge, it is necessary to make a test to find whether the edge is straight. Draw a line along the edge, as AB, with the ruler below the line, in the position $ABDC$. Then place the ruler above the line in the position $ABFE$, so that C falls on E, and D falls on F. The edge is straight if a second line drawn from A to B coincides with the first line.



226. Segment of a Line. Any part of a line is a **segment** of that line. It is often called a **line-segment.**



Thus EF and HK are segments of the line AB.

227. Broken Line. When a line is made up of two or more connected line-segments, as in the figure, it is a **broken line.**

Thus BD is a broken line.

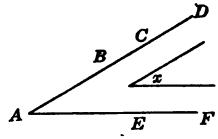
228. Plane Figures. Geometric figures, all points of which are in the same plane, are **plane figures.**

In Plane Geometry, wherever the word figure is used, it should be remembered that plane figure is understood.

229. Rectilinear Figures. Plane figures bounded by straight lines are **rectilinear figures.**

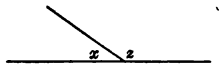
230. Angle. The extent of divergence of two lines extending from the same point is an **angle.**

231. Size of Angles. The size of an angle depends only upon the extent of divergence of its sides and is independent of the length of its sides or the amount of space between them.



Thus $\measuredangle BAE$, CAE, BAF, and DAF are the same angle. $\angle x$ is the same size as $\angle A$.

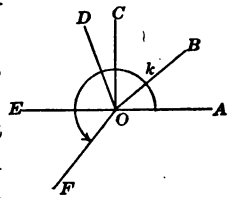
232. Supplementary Adjacent Angles. If two angles are both supplementary and adjacent, they are **supplementary adjacent angles.**



Thus $\measuredangle x$ and z are supplementary adjacent angles.

233. Oblique Angles and Lines. Acute angles and obtuse angles may be called **oblique angles.** Intersecting lines which are not perpendicular to each other may be called **oblique lines.** Lines forming oblique angles are oblique lines.

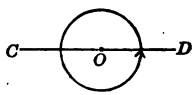
234. Generation of Angles. If the line k moves from coincidence with OA about the point O to the position OE, the straight angle AOE is formed.



If the line k moves to the position OF, the **reflex angle** AOF is formed.

If the line k moves about O until it returns to the position OA, it forms an angle of complete rotation, or a **perigon.**

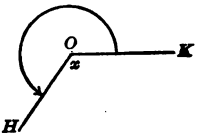
235. Zero Angle and Perigon. If the sides of an angle extend in *opposite directions* so that they form a straight line, the angle is a **straight angle.**



If the sides of an angle extend in the *same direction* so that they form a straight line, the angle may be either a **zero angle** or a **perigon.**

Suppose the line OC is placed upon the line OD. Before they have been moved, they may be considered to form a **zero angle.** After OC has made a complete rotation about O, and again coincides with OD, the two lines may be considered to form a **perigon,** which is equal to two straight angles, or 360°.

236. Reflex Angle. An angle which is greater than a straight angle but less than a perigon or two straight angles is a **reflex angle.**



Thus $\angle KOH$ is a reflex angle.

237. Conjugate Angles. Two angles whose sum is a perigon, or 360°, are called **conjugate angles.**

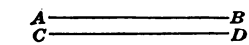
The reflex angle KOH and $\angle x$ are conjugate angles.

238. Coincide. To *coincide* is to *fall together* or to occupy the same place in space.

239. Test of Equality. If two geometric magnitudes can be made to coincide throughout their whole extent, they are equal.

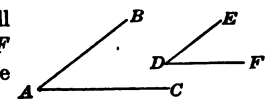
240. Equality of Lines. *Two straight lines are equal if their extremities can be made to coincide when one line is placed upon the other.* § 66

Thus CD is equal to AB if C and D fall on A and B respectively when CD is placed on AB.



241. Equality of Angles. *Two angles are equal if the sides of one can be made to fall along the sides of the other, when their vertices are made to coincide.*

Thus $\angle A = \angle D$, if AB can be made to fall along DE, and AC can be made to fall along DF when the vertex A is made to coincide with the vertex D.



242. Corresponding sides and **corresponding angles** of two triangles are the sides and angles which occupy *similar relative positions* in the two triangles. In two equal triangles, the equal angles are called *corresponding angles*, and the equal sides are called *corresponding sides.*

243. Corollary. A truth which is easily deduced from known truths is sometimes called a **corollary.**

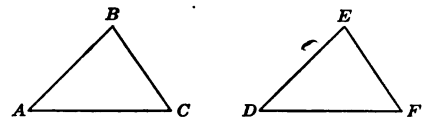
For example, the theorem of § 90 might be called a corollary of § 89. Also §§ 144 and 145 are corollaries of § 143.

244. Proof. A theorem is said to be *proved* when its truth is based upon other truths which have been previously established or accepted.

METHODS OF PROOF

245. Superposition. Two geometric figures may be shown to be equal or unequal by **superposition**; that is, by properly placing one upon the other.

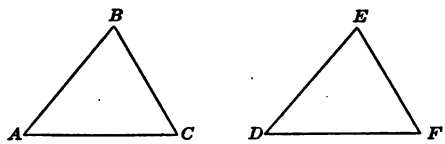
Thus, $\triangle ABC$ is equal to $\triangle DEF$, if they coincide throughout, when one is placed upon the other.



In *theoretical superposition*, the process is *imagined* to be done, as in the following theorems.

246. Congruence. If two geometric figures can be shown to coincide by superposition, they are equal in the sense that they are **equal in every respect**, or **congruent.** Figures which are congruent have the same size and the same shape.

247. Theorem. ***Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.***



Given $\triangle ABC$ and DEF, having $AB = DE$, $AC = DF$, and $\angle A = \angle D$.

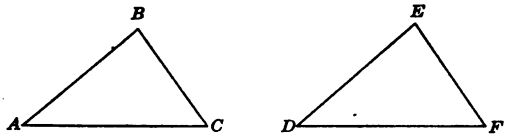
To prove $\triangle ABC \cong \triangle DEF$.

Proof. Place $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ falls on $\angle D$ and AC falls along DF.

The point C will fall on the point F.
(Since it is given that $AC = DF$.)
AB will fall along DE.
(Since it is given that $\angle A = \angle D$.)

Then	the point B will fall on the point E.	Why?
And	BC will coincide with EF.	§ 240
	$\therefore \triangle ABC \cong \triangle DEF$.	§ 246

248. Theorem. ***Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.***



Given $\triangle ABC$ and DEF, having $\angle A = \angle D$, $\angle C = \angle F$, and $AC = DF$.

To prove $\triangle ABC \cong \triangle DEF$.

Proof. Place $\triangle ABC$ upon $\triangle DEF$ so that AC falls on DF.

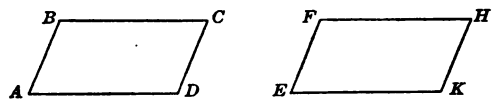
Then AB will fall along DE, since $\angle A = \angle D$.

Also BC will fall along EF, since $\angle C = \angle F$.

Since two straight lines can intersect in only one point, B must fall on E.

$$\therefore \triangle ABC \cong \triangle DEF. \qquad \S\ 246$$

249. Theorem. *Two parallelograms are congruent if they have two adjacent sides and the included angle of one equal respectively to two adjacent sides and the included angle of the other.*



Given parallelograms $ABCD$ and $EFHK$, having $AB = EF$, $AD = EK$, and $\angle A = \angle E$.

To prove $\square ABCD \cong \square EFHK$.

Proof. Place $\square ABCD$ upon $\square EFHK$ so that AD coincides with EK.

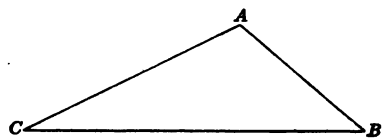
Then	AB will coincide with EF.	Why?
	BC will fall along FH.	Why?
And	DC will fall along KH.	
Then	Point C will coincide with point H.	Why?

$$\therefore \square ABCD \cong \square EFHK.$$

250. Indirect Proof. When the indirect proof of a proposition is used, the line of reasoning begins by supposing that the conclusion of the proposition is wrong. Then it is shown that this supposition leads to a statement which is not true. This proves that the supposition is false and therefore, that the original proposition is true.

This method of proof is known as the *reductio ad absurdum* (reduction to an absurdity).

251. Theorem. *If two angles of a triangle are unequal, the sides opposite these angles are unequal, the greater side being opposite the greater angle.*



Given $\angle A > \angle C$ in $\triangle ABC$.

To prove $BC > BA$.

Proof. It must be true that $BC = BA$ or $BC < BA$ or $BC > BA$.

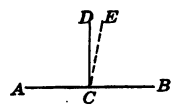
Suppose $BC = BA$. Then $\angle A = \angle C$. Why?

Suppose $BC < BA$. Then $\angle A < \angle C$. Why?

But, since it was given that $\angle A > \angle C$, neither of these suppositions can be true.

$$\therefore BC > BA.$$

252. Theorem. *Only one perpendicular can be drawn to a given line at a given point in the line.*



Given $DC \perp AB$ at C.

To prove that no other perpendicular can be drawn to AB at C.

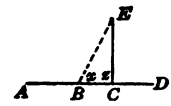
Proof. Suppose another line $EC \perp AB$ at C.

Then $\angle ECB$ and $\angle DCB$ are both right angles.

But, since all right angles are equal, $\angle ECB$ and $\angle DCB$ cannot both be right angles.

Therefore the supposition that $EC \perp AB$ is not true and DC is the only perpendicular which can be drawn to AB at C.

253. Theorem. *Only one perpendicular can be drawn to a given line from a given external point.*



Given $EC \perp AD$ from E.

To prove that no other perpendicular can be drawn from E to AD.

Proof. Suppose another line $EB \perp AD$ from E.

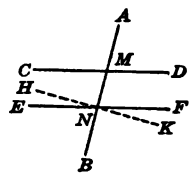
Then $\angle x$ and $\angle z$ are both right angles. Why?

But this is impossible since a triangle can have only one right angle.

Then the supposition that $EB \perp AD$ is not true.

$\therefore$ EC is the only perpendicular that can be drawn to AD from E.

254. Theorem. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Given CD and EF, two parallel lines cut by the transversal AB.

To prove $\angle CMN = \angle MNF$.

Proof. Suppose $\angle CMN$ is not equal to $\angle MNF$.

Also suppose a line HK to be drawn through N, making $\angle MNK = \angle CMN$.

Then $HK \parallel CD$ Why?

But $EF \parallel CD$. Why?

Then, through N, there are two lines parallel to CD, which is impossible. Why?

Hence the supposition that $\angle CMN$ is not equal to $\angle MNF$ is not true.

$\therefore$ $\angle CMN = \angle MNF$.

255. Theorem. *If two parallel lines are cut by a transversal, the exterior interior angles are equal.*

Prove by the indirect method, using the figure of § 254.

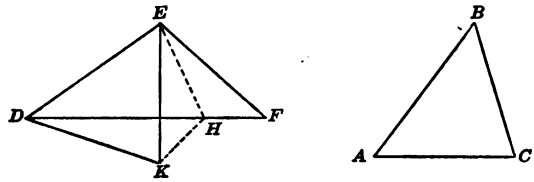
256. Theorem. *If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.*

Prove by the indirect method.

257. Theorem. *If two parallel lines are cut by a transversal, the alternate exterior angles are equal.*

Prove by the indirect method.

258. Theorem. *If two triangles have two sides of one equal respectively to two sides of the other but the included angle of the first triangle greater than the included angle of the second, then the third side of the first is greater than the third side of the second*



Given triangles DEF and ABC having $DE=AB$, $EF=BC$, and $\angle E>\angle B$.

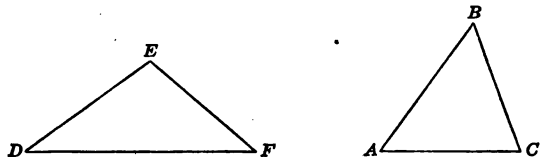
To prove that $DF>AC$.

Proof. Place $\triangle ABC$ upon $\triangle DEF$ so that AB coincides with DE and $\triangle ABC$ takes the position of $\triangle DEK$.

Bisect $\angle KEF$ by the line EH and draw KH.

	$\triangle HEK=\triangle HEF$	Why?
Then	$HK=HF$	Why?
But	$DH+HK>DK$	Why?
Then	$DH+HF>DK$	Why?
Or	$DF>DK$	
	$\therefore DF>AC$	Why?

259. Theorem. *If two triangles have two sides of one equal to two sides of the other but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given the triangles DEF and ABC having $DE=AB$, $EF=BC$ and $DF>AC$.

To prove $\angle E>\angle B$.

Proof. It must be true that $\angle E=\angle B$ or $\angle E<\angle B$ or $\angle E>\angle B$.

Suppose $\angle E=\angle B$. Then $DF=AC$. Why?

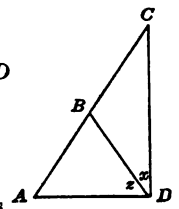
Suppose $\angle E<\angle B$. Then $DF<AC$. § 258

Since it was given that $DF>AC$, neither of these suppositions can be true. $\therefore \angle E>\angle B$.

260. Proof by Analysis. When the proof of an exercise is not readily seen, the figure should be analyzed to find conditions under which the conclusion must be true. This may be illustrated by the following exercise.

EXERCISE

If the middle point of one side of a triangle is equidistant from the three vertices, the triangle is a right triangle.



Given that B is equidistant from A, C, and D.

To prove that $\triangle ACD$ is a right triangle.

Analysis and proof. $\triangle ACD$ is a right triangle if $\angle D$ is a right angle.

$\angle D$ is a right angle if $\angle D=\angle A+\angle C$.

$\angle D=\angle A+\angle C$ if $\angle z+\angle x=\angle A+\angle C$.

$\angle z+\angle x=\angle A+\angle C$ if $\angle z=\angle A$ and $\angle x=\angle C$.

$\angle z=\angle A$ and $\angle x=\angle C$ if $\triangle ABD$ and CBD are isosceles.

$\triangle ABD$ and CBD are isosceles if $AB=BD$ and $BC=BD$.

But $AB=BD$ and $BC=BD$ since it is given that B is equidistant from A, C, and D.

Therefore all the conditions are true and $\triangle ACD$ is a right triangle.

261. Proof by Synthesis. By the **synthetic method** of proof, a truth is proved by putting together other truths which have previously been proved or established. This is the method most commonly used. Nearly all of the theorems and exercises in Chapter I are proved by this method.

In many exercises the proof is almost obvious from an examination of the figure and the hypothesis. In more difficult exercises, it is well to resort to analysis in order to discover the proof.

QUESTIONS

1. If one of the acute angles of a right triangle is given, how can the other be found?

2. If one angle of a triangle is a right angle, what relation exists between the other two angles?

3. How many altitudes has a triangle? How many medians?

4. How many right angles can there be in a triangle? How many obtuse angles? How many acute angles?

5. How many degrees in each angle of an equilateral triangle?

6. A right triangle is isosceles. How many degrees in each angle?

7. A right triangle has acute angles of 30° and 60°. What is the relation between the hypotenuse and the shorter side?

8. What does the altitude from the vertex of an isosceles triangle do to the base? To the vertex angle? To the triangle?

9. State several facts about the altitudes of an equilateral triangle.

10. Where do the three medians of any triangle meet?

11. In what quadrilaterals are the diagonals perpendicular to each other?

12. In what quadrilaterals do the diagonals bisect each other? In what ones do they bisect the angles?

13. State as many facts as you can about the diagonals of a square. Of a rhombus. Of a rectangle.

14. State different methods of proving two triangles equal.

15. State methods of proving two right triangles equal.

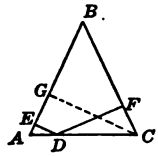
16. Give several conditions which show that two lines are equal.

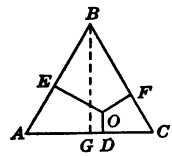
17. Give several conditions which show that two angles are equal.

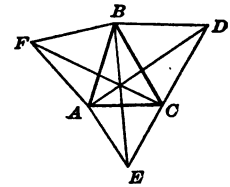
18. What are some of the methods of proving that two lines are parallel?

EXERCISES INVOLVING EQUAL LINES

1. The sum of the perpendiculars from any point in the base of an isosceles triangle to the two equal sides equals the perpendicular from either end of the base to the opposite side.



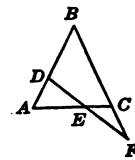




2. If from any point in an equileteral triangle perpendiculars are drawn to the three sides, the sum of these perpendiculars is constant, and equal to the altitude of the triangle.

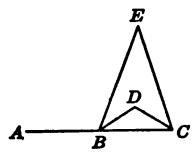
3. If upon the three sides of any triangle, equilateral triangles are constructed outward, and a line is drawn from each vertex of the triangle to the farthest vertex of the equilateral triangle on the opposite side, these three lines are equal. Prove that $AD = FC = BE$.

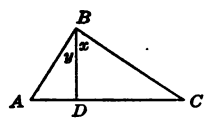
4. The difference of the perpendiculars from any point in the base produced of an isosceles triangle to the equal sides equals the perpendicular from either end of the base to the opposite side.

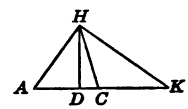


5. Triangle ABC is isosceles. $AD = CF$. Prove that DF is bisected by the base AC.

EXERCISES INVOLVING EQUAL ANGLES



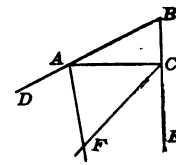




1. Lines BD and CD bisect the base angles of the isosceles triangle BEC. Prove that $\angle ABE = \angle BDC$.

2. In right triangle ABC, $DB \perp AC$. Prove that $\angle x - \angle y = \angle A - \angle C$.

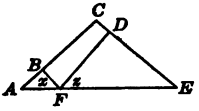
3. Figure AHK is a right triangle with $\angle H$ the right angle. HD is an altitude and HC a median. Prove that $\angle DHC = \angle A - \angle K$, and that $\angle AHD = \angle CHK$.



4. The angle formed by the bisectors of two exterior angles of a triangle equals half the sum of the interior angles at the same vertices.

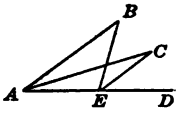
5. In a right triangle, prove that the angle formed by the bisector of the right angle and the median drawn to the hypotenuse equals half the difference of the two acute angles.

6. In the isosceles triangle ACE, FB and FD are perpendicular to the equal sides. Prove that $\angle x = \angle z$.



7. In a right triangle, the angle formed by the median and the altitude drawn to the hypotenuse is bisected by the bisector of the right angle.

8. In the figure, AC bisects $\angle BAE$, and EC bisects $\angle BED$. Prove that $\angle C = \frac{1}{2}\angle B$.



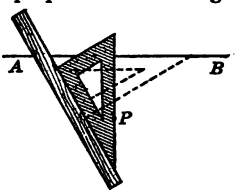
EXERCISES INVOLVING PARALLELS AND PERPENDICULARS

1. If one of the equal sides of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the side produced to the nearest end of the base is perpendicular to the base.

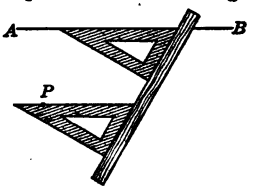
2. If medians are drawn to the equal sides of an isosceles triangle, the line joining the points of meeting is parallel to the base.

3. If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.

4. Prove that the following method of drawing a perpendicular through P to AB is correct. First, place the triangle in the position shown by the dotted lines, with its hypotenuse along AB. Second, place a straightedge along the side of the triangle, as shown, and hold it in that position. Third, place the triangle in the position shown by the shaded triangle. Fourth, draw a line through P along the hypotenuse of the triangle. This is the perpendicular required.



5. Prove that the following method of drawing a parallel to AB through P is correct. First, place the triangle with the hypotenuse along the line AB. Second, place a straightedge along a side of the triangle, and hold in that position while the triangle is placed so that its hypotenuse passes through P. The line through P along the hypotenuse is parallel to AB.



EXERCISES INVOLVING INEQUALITIES

1. Prove that the sum of the sides of a quadrilateral is greater than the sum of its diagonals, and less than twice their sum.

2. Prove that the difference between the diagonals of a quadrilateral is less than the sum of either pair of opposite sides.

3. If one side of a triangle is produced in both directions, the sum of the exterior angles is greater than two right angles.

4. The sum of the three altitudes of a triangle is less than the perimeter.

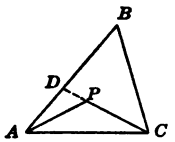


FIG. 1

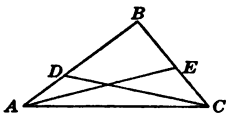


FIG. 2

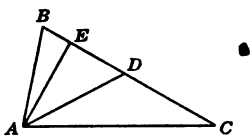


FIG. 3

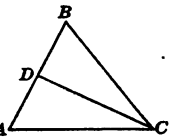
5. If P is any point within the triangle ABC (Fig. 1), prove that $\angle APC > \angle B$.

6. In the triangle ABC (Fig. 2), $AB > BC$. AD is taken equal to CE. Prove that $AE > CD$.

7. In triangle ABC (Fig. 3), $AC > AB$, $AE \perp BC$, and AD is a median. Prove that: (1) $\angle CAE > \angle BAE$; (2) $\angle CAD < \angle BAD$; (3) the bisector of $\angle BAC$ falls between AE and AD.

8. Draw a quadrilateral $ABCD$ having $\angle B = \angle C$, and $AB < CD$. Prove that $\angle A > \angle D$.

9. In any triangle ABC, angle C is bisected, the bisector meeting AB at D. Prove that $BC > BD$, and that $AC > AD$.

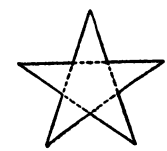


10. If from any point within a triangle lines are drawn to the extremities of the sides, the sum of these lines is less than the sum of the sides of the triangle.

COMPUTATION EXERCISES

1. What is the size of the obtuse angle formed by the bisectors of the acute angles of any right triangle?

2. Find the value of the sum of the angles at the vertices of a regular five-pointed star. (Such a star is called a pentagram. The pentagram-star was used by the Pythagoreans as a badge, and among them it stood for Health.)



3. What part of its complement is an angle that is one-fourth of its supplement? One-fifth? One-ninth?

4. What part of its supplement is an angle that is four times its complement? Five times? Six times?

5. Find the values of the three angles of a triangle if the second is three times the first and the third is twice the second.

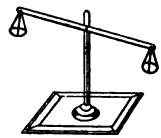
6. The perimeter of an isosceles triangle is 8 ft. 2 in., and the base is one-sixth the sum of the equal sides. Find the lengths of the three sides.

7. A rectangle that is three times as long as wide has a perimeter of 6 ft. Find its dimensions in inches.

8. A rectangle is 10 in. longer than wide. Find its dimensions if its perimeter is 10 ft. 8 in. *Ans.* 37 in. by 27 in.

MISCELLANEOUS EXERCISES

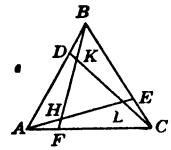
1. Prove that in balance scales the scale pans are always the same distance from the middle post.



2. Prove that the three altitudes of an equilateral triangle are equal. Prove that they are also the medians of the triangle, the perpendicular bisectors of the sides of the triangle, and the bisectors of the angles of the triangle.

3. If two parallel lines are cut by a transversal, the bisectors of the four interior angles form a rectangle.

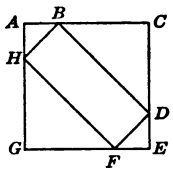
4. Triangle ABC is equilateral, and $BD = AF = CE$. Prove that $\triangle HKL$ is equilateral.



5. $ACEG$ is a square in which $AB = AH = ED = EF$. Prove that $HBDF$ is a rectangle.

6. Two isosceles triangles are congruent if they have the following corresponding parts equal:

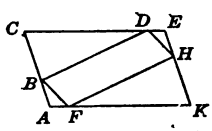
(1) Base and vertex angle.
(2) Base and one of the adjoining angles.
(3) One of the equal sides and a base angle.
(4) One of the equal sides and vertex angle.
(5) One of the equal sides and the base.



7. Two triangles are congruent if they have a side and the altitude and median to that side in one equal to corresponding parts in the other.

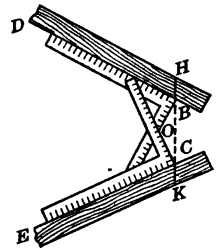
8. If the median of a triangle equals one-half the side it bisects, the triangle is a right triangle.

9. In the figure, $ACEK$ is a parallelogram. $AF = ED$, and $BC = HK$. Prove that $BDHF$ is a parallelogram.



10. Show how two converging timbers may be cut on a line which makes equal angles with the timbers. Two carpenter's squares are placed as in the figure, so that $OB = OC$.

Prove that $\angle DBC = \angle ECB$.

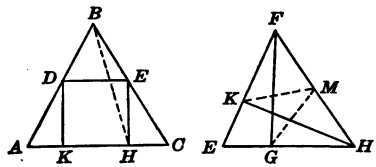


11. In an isosceles trapezoid, show that the triangles formed by the two diagonals and the bases are isosceles.

12. Triangle ABC is equilateral. Show how a square $DEHK$ may be constructed within the triangle by first making $\angle HBC = 15°$.

13. In the figure, $\triangle EFH$ is any triangle with M the middle point of FH. Two altitudes of the triangle are FG and HK.

Prove that $MK = MG$. § 193

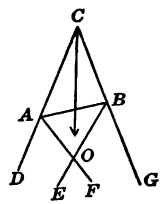


14. The bisectors of the angles of a rhomboid form a rectangle.

15. The bisectors of the angles of a rectangle form a square.

16. The lines which join the middle points of the sides of a rhombus, taken in order, form a rectangle.

17. The lines which join the middle points of the sides of an isosceles trapezoid, taken in order, form a rhombus.



18. The bisectors of two exterior angles of a triangle and of the interior angle at the third vertex are concurrent at a point equally distant from the three sides.

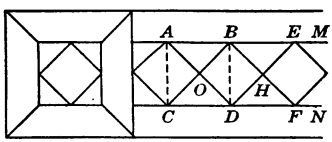


FIG. 1

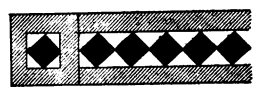
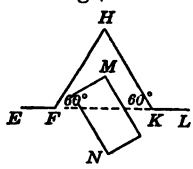
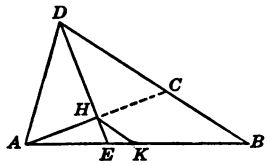


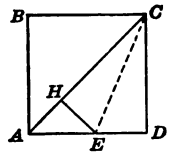
FIG. 2

19. Fig. 2 is a parquet floor border, the outline of which is in Fig. 1. On the two parallel lines AM and CN, segments are laid off so that $AC = AB = CD = BE = DF$. Show that: (1) $ABDC$ is a square; (2) $\triangle COD$ is an isosceles right triangle; (3) $OBHD$ is a square.

20. Show how a line EF may be produced beyond a building MN, as in this figure.



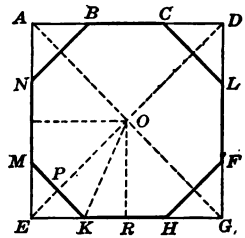




21. In $\triangle ADB$, the line DE bisects $\angle ADB$, $AH \perp DE$, and K is the middle point of AB. Prove that $HK = \frac{1}{2}(DB - DA)$. Also prove that $\angle CAB = \frac{1}{2}(\angle DAB - \angle B)$.

22. In the figure, $ABCD$ is a square. On the diagonal AC, CH is laid off equal to CD, and EH is drawn perpendicular to AC to meet AD in E. Prove that $AH = EH = ED$.

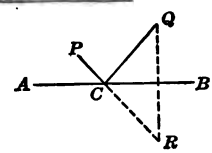
23. In this figure, $PK = KR$ as in the preceding Exercise. Construct MK and similarly NB, CL, and FH. Show that $BCLFHKMN$ is a regular octagon. Also show that $EO = EH$. Thus show that a regular octagon can be constructed in a square by laying off along the sides from each vertex, lengths equal to half the diagonal of the square, and in this way determine the vertices of the octagon.



24. Construct a square. Within this construct a regular octagon in accordance with the directions given in the preceding Exercise. Apply to the construction of an octagonal top for a table.

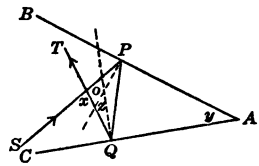
25. Construct the geometric forms in the above steel ceiling designs, and sketch the designs.

26. P and Q are two points on the same side of a line AB. Find the position of a point C in AB such that $\angle PCA = \angle QCB$.



27. Show that the sum of three exterior angles A, C, and E in any hexagon, $ABCDEF$, is 180° less than the sum of the three interior angles at B, D, and F.

28. If AB and AC are two mirrors placed so as to form an acute angle with each other, and if SP is a ray of light striking AB so as to be reflected first to Q and next to T, the angle x equals twice angle y.

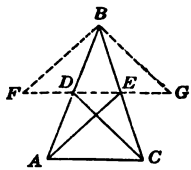


SUGGESTION. Prove that $\angle x = 2\angle z$, and $\angle z = \angle y$.

29. If two altitudes of a triangle are equal, the triangle is isosceles.

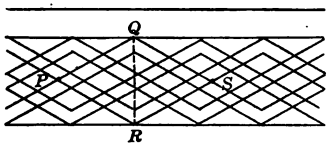
30. If two medians of a triangle are equal, the triangle is isosceles.

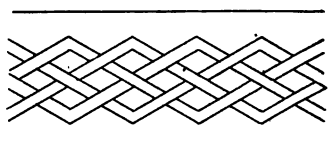
SUGGESTION. Use $\triangle ABC$ with construction lines as drawn.



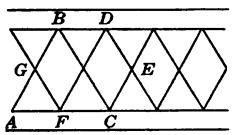
31. The double equilateral triangle known as the "diamond" enters into many designs for parquet floor borders, tile designs, etc.

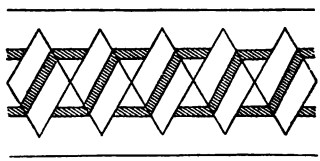
Construct the design shown in the figure, which is a parquet floor border, depending upon the "diamond." How many degrees in the angle PQS?

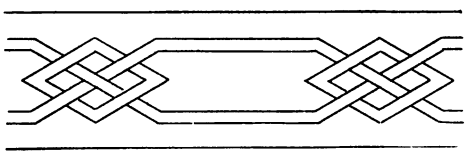




32. Construct the parquet floor borders shown in the following figures. Note that $BDECFG$ is a regular hexagon.





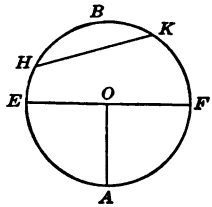


CHAPTER II

THE CIRCLE

262. Curved Line. A line which changes its direction at every point is a **curved line.** A *curved line* is a line no part of which is straight.

263. Circle. A **circle** is a closed curved line all points of which are in the same plane and equally distant from a point within, called the **center.**



Thus *EKA* is a circle. The circle may also be read, "the circle *O*."

The line forming the circle is sometimes called the circumference.

264. Arcs. If a circle is divided into two unequal arcs, the larger is called the **major arc,** and the smaller is called the **minor arc.**

Thus *HAK* is a major arc and *HBK* is a minor arc.

A **semicircle** is an arc which is one-half of a circle. A **quadrant** is an arc which is one-fourth of a circle.

The arcs *EAF* and *EKF* are semicircles. The arcs *EA* and *AF* are quadrants.

265. Equal Radii. Since all points on the circle are equidistant from the center, *radii of the same circle, or of equal circles, are equal.* Also, *circles having equal radii are equal.*

266. Chord. A straight line having its ends in the circle is a **chord.**

Thus *HK* is a chord. The chord *HK* is said to **subtend** the arc *HK*. It also subtends the arc *HAK*. The arc *HK* is **subtended** by the chord *HK*. To subtend is to *stretch under*.

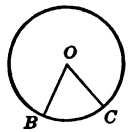
267. Diameter. A chord passing through the center of a circle is a **diameter.**

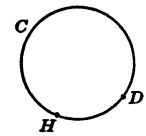
Thus *EF* is a diameter.

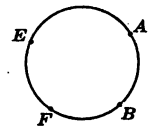
268. Since a diameter is always made up of two radii, *diameters of the same circle, or of equal circles, are equal.*

269. Central Angle. An angle formed by two radii is a **central angle.**

Thus $\angle BOC$ is a central angle.







270. The circles CDH and EAB are equal circles.

If any arc, as AB, is placed with its ends upon either circle, at D and H or at E and F, so that it curves the same as the circle, the entire arc must fall upon the circle, since all points on the circles are equidistant from the centers.

271. Experiment. The circles below have a radius of 2 cm. Angles x and z are each 65°. On a piece of heavy paper construct a circle with a radius of 2 cm. Make a central angle z of 65°. Letter this angle ABC. Cut out the portion of the circle included between the radii BA and BC.

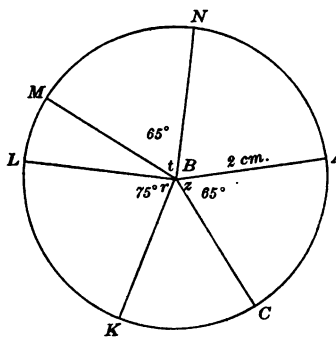


Fig. 1

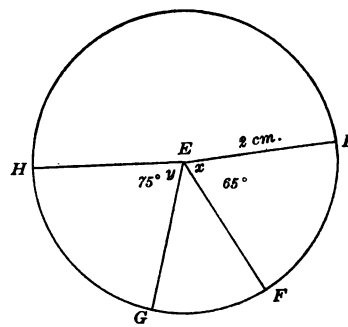


Fig. 2

Place the angle ABC successively upon the angles x, y, r, and t so that BA coincides with ED, EG, BK, and BM. Which of the arcs DF, GH, KL, and MN does the arc AC equal?

This experiment shows that:

272. ***Equal central angles in the same circle or in equal circles intercept equal arcs.*** Conversely: ***Equal arcs are intercepted by equal central angles.***

273. *Of two unequal central angles, in the same circle or in equal circles, the greater central angle intercepts the greater arc.*

Conversely: *Of two unequal arcs, the greater arc is intercepted by the greater central angle.*

274. These statements may be proved by theoretical superposition. To do this place $\odot B$ upon $\odot E$ so that BC coincides with its equal, EF. Then show that BA must coincide with ED and that $\overset{\frown}{CA}$ must coincide with $\overset{\frown}{FD}$.

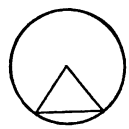
EXERCISES

1. Where are all points two inches from a given point?

2. Where are all points within one mile of the Post Office?

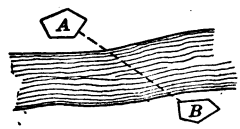
3. On a warship, a cannon which shoots ten miles brings into danger all persons within a radius of ten miles. Explain.

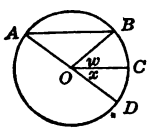
4. In opening a door, what kind of line is generated by a corner of the door?

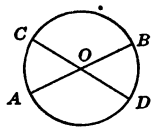


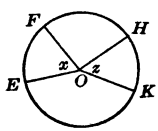
5. Radii drawn to the ends of a chord form with the chord an isosceles triangle.

6. Two forts at A and B are 11 miles apart on opposite sides of a river as in the figure. Show by a drawing, whether or not the river is completely fortified by guns which throw shells five miles.









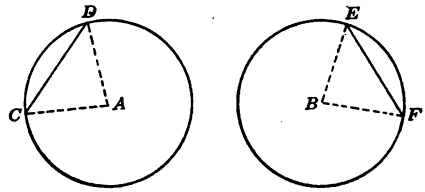
7. Chord AB is parallel to radius OC. Prove that $\overset{\frown}{BC} = \overset{\frown}{CD}$.

8. If AB and CD are diameters of the circle, prove that $\overset{\frown}{AC} = \overset{\frown}{DB}$ and $\overset{\frown}{AD} = \overset{\frown}{CB}$.

9. If $\angle x = \angle z$, prove that arc EFH = arc FHK.

ARCS AND CHORDS

275. Theorem. *In the same circle or in equal circles, equal chords subtend equal arcs.* Conversely: *Equal arcs are subtended by equal chords.*



Given the equal circles with centers A and B. Chord CD = chord EF.

To prove $\overset{\frown}{CD} = \overset{\frown}{EF}$.

Proof. Draw the radii AD, AC, BF, and BE.

	$BE = AC.\quad BF = AD.$	§ 265
	$\triangle ADC = \triangle BFE.$	Why?
Then	$\angle A = \angle B.$	Why?
	$\therefore \overset{\frown}{CD} = \overset{\frown}{EF}.$	§ 272

CONVERSELY:

Given $\overset{\frown}{CD} = \overset{\frown}{EF}$.

To prove chord CD = chord EF.

Proof.

	$\angle A = \angle B.$	§ 272
	$\triangle ADC = \triangle BFE.$	Why?
	$\therefore$ chord CD = chord EF.	Why?

EXERCISES

1. In Fig. 1, EF is a diameter of the circle. Radius $OB \perp EF$. $\angle x = \angle z$. Prove that chord DB = chord CB.

2. The chords DB and FE (Fig. 2) are equal. Prove that $DE = FB$, and $\angle D = \angle F$.

3. In a circle, draw two perpendicular diameters, AB and CD. Join AD, AC, BD, and BC. Prove that $ADBC$ is a square.

4. In a circle, draw two intersecting diameters, AB and CD, which are not perpendicular to each other. Draw the chords AC, AD, BC, and BD. Prove that $ADBC$ is a rectangle.

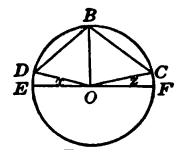


FIG. 1

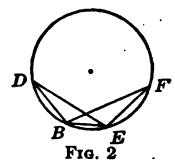
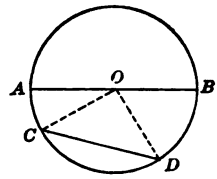


FIG. 2

276. Theorem. *A diameter of a circle is greater than any other chord.*

Given AB, a diameter of the circle whose center is O, and CD any other chord.

To prove $AB > CD$.



Proof. Draw the radii OC and OD.

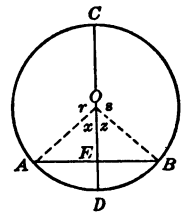
$OC + OD > CD$. Why?

Then $AO + OB > CD$. Why?

$\therefore AB > CD$.

277. Theorem. ***A diameter perpendicular to a chord bisects the chord and the arc subtended by it.***

Given the circle whose center is O, with the diameter CD perpendicular to the chord AB.



To prove $AE = EB$ and $\overset{\frown}{AD} = \overset{\frown}{DB}$.

Proof. Draw the radii OA and OB.

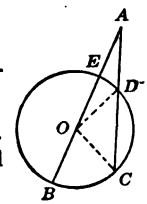
$OA = OB$. Why?

Show that $\triangle OEA = \triangle OEB$.

Then $AE = EB$, and $\angle x = \angle z$. Why?

$\therefore \overset{\frown}{AD} = \overset{\frown}{DB}$. § 272

EXERCISES



1. In the figure of § 277, prove that $\overset{\frown}{AC} = \overset{\frown}{BC}$.

2. Through a point within a circle draw the longest possible chord.

3. In the circle O, BE is a diameter and CD is a chord. Both are produced to meet at A. Prove that $BA > CA$ and $EA < DA$.

278. Theorem. *A diameter bisects the circle.*

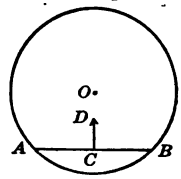
279. Theorem. *A diameter which bisects a chord is perpendicular to the chord.*

Given the diameter CD which bisects the chord AB.

To prove $CD \perp AB$.

(Complete the proof.)

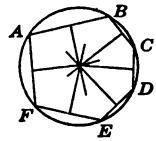
280. Theorem. *The perpendicular bisector of a chord passes through the center of the circle.*



CD is the perpendicular bisector of chord AB.
O is equidistant from A and B. § 263
$\therefore$ O is in the line CD produced, and CD must pass through O. § 206

EXERCISES

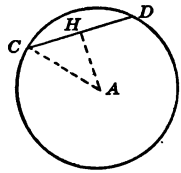
1. The sides of the polygon $ABCDEF$ are chords of the circle. Show that the perpendicular bisectors of the sides pass through the same point.

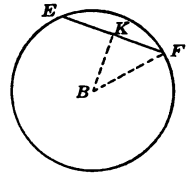


2. In a circle draw five parallel chords. Are the centers of these chords in the same straight line? Why? Where are the centers of all parallel chords in a circle?

3. Locate a point within a given circle. Through this point draw a chord which shall be bisected at that point.

281. Theorem. *In the same circle or in equal circles, equal chords are equidistant from the center.* Conversely: *Chords equidistant from the center are equal.*





Given chord CD equal to chord EF in the equal circles A and B.

To prove that CD and EF are equidistant from A and B.

Proof. Draw the radii AC and BF. Also draw $AH \perp CD$, and $BK \perp EF$.

Then AH bisects CD and BK bisects EF. § 277
Show that $\triangle AHC = \triangle BKF$. Then $AH = BK$. Why?
$\therefore$ CD and EF are equidistant from the centers. § 189

CONVERSELY:

Given $AH = BK$ in the equal circles A and B.

To prove chord CD = chord EF.

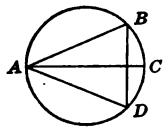
Proof.
$\triangle AHC = \triangle BKF$. Why?
Then $CH = KF$. Why?
$\therefore$ $CD = EF$. Why?

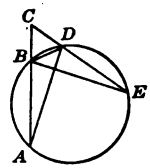
EXERCISES

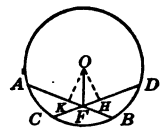
1. Where are the middle points of all equal chords of a circle?

2. The diameter AC is the perpendicular bisector of the chord BD. Prove that $\angle B = \angle D$.

3. Chord AB = chord ED. Prove that $\triangle CBD$ is isosceles.







4. Two equal chords intersect within a circle. Prove that the corresponding segments are equal.

5. If two intersecting chords make equal angles with the diameter drawn through the point of intersection, the chords are equal.

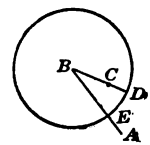
6. Two equal chords intersect within a circle. The center is joined to the point of intersection by the line OF. Prove that OF makes equal angles with each of the chords.

7. If two chords intersect so that a segment of one is equal to a segment of the other, the chords are equal.

8. Through a given point within a circle, draw two equal chords.

9. If two chords are bisected by a diameter of the circle, prove that the chords are parallel.

282. Since all points on a circle are equidistant from the center, *the distance from the center of a circle to any point without the circle is greater than the radius, and the distance from the center to any point within the circle is less than the radius.*



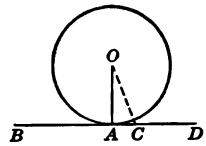
$BA >$ radius BE and $BC <$ radius BD. § 110

283. *A point is without a circle when its distance from the center is greater than the radius.* Also, *a point is within a circle when its distance from the center is less than the radius.*

These must be true from the definition of a circle. § 263

TANGENTS AND SECANTS

284. Theorem. *A perpendicular to a radius at its outer extremity can touch the circle at only one point.*



Given BD perpendicular to the radius OA at its outer extremity A.

To prove BD can touch the circle at only one point.

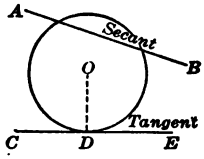
Proof. Draw OC, any other line, from the center O to the line BD.

Then $OC > OA$ (§ 185), and the point C must lie without the circle. § 283

Hence every point in the line BD, except A, lies without the circle.

$\therefore$ BD can touch the circle at only one point.

285. Tangent. A straight line which touches a circle at only one point is a **tangent.**



Thus CE is a tangent to the circle O. The circle is also said to be tangent to the line CE. The point D is called the *point of contact* or *point of tangency*. In speaking of a *tangent to a circle from an external point*, only that part of the tangent is considered which lies between the external point and the point of contact. CD is the tangent to the circle O from the point C.

286. Secant. A straight line which intersects a circle in two points is a **secant.**

In the figure, AB is a secant.

287. Theorem. ***A straight line perpendicular to a radius at the point where the radius meets the circle is tangent to the circle.*** § 284 and § 285

288. Theorem. *The radius drawn to the point of tangency is perpendicular to the tangent.*

Since only one radius can be drawn to a tangent, this must be true from § 287.

EXERCISE

Tangents to a circle at the extremities of a diameter are parallel.

289. Theorem. *Parallel lines intercept equal arcs on a circle.*

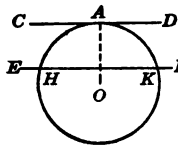


FIG. 1

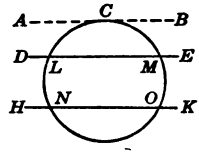


FIG. 2

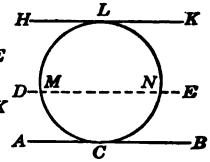


FIG. 3

CASE I. When a tangent and a secant are parallel. (Fig. 1.)

Given the secant EF parallel to the tangent CD.

To prove $\overset{\frown}{AH} = \overset{\frown}{AK}$.

Proof. Draw the radius OA to the point of contact.

Then	$CD \perp OA$.	§ 288
Also	$EF \perp OA$.	§ 123
	$\therefore \overset{\frown}{AH} = \overset{\frown}{AK}$.	§ 277

CASE II. When two secants are parallel. (Fig. 2.)

Given the parallel secants DE and HK.

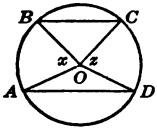
To prove $\overset{\frown}{LN} = \overset{\frown}{MO}$.

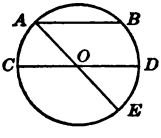
Proof. Draw the tangent $AB \parallel DE$.

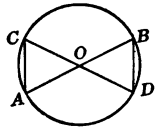
Then	$AB \parallel HK$.	§ 124
	$\overset{\frown}{CN} = \overset{\frown}{CO}$ and $\overset{\frown}{CL} = \overset{\frown}{CM}$.	Case I
	$\therefore \overset{\frown}{LN} = \overset{\frown}{MO}$.	§ 106

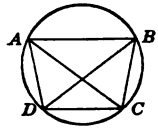
CASE III. When two tangents are parallel. (Fig. 3.)
(Complete the proof.)

EXERCISES



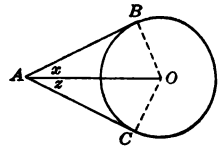






1. The lines BC and AD are parallel chords. Prove that $\angle x = \angle z$.
2. CD and AE are intersecting diameters. $AB \parallel CD$. Prove $\overset{\frown}{BD} = \overset{\frown}{DE}$.
3. If AB and CD are intersecting diameters, prove that $AC \parallel BD$.
4. $AB \parallel DC$. Prove that $ABCD$ is an isosceles trapezoid.
5. Show that the diagonals of an isosceles trapezoid are equal.

290. Theorem. *Tangents to a circle from an external point, are equal, and make equal angles with the line joining that point with the center.*



Given AB and AC, tangents to the circle O, from the external point A, and AO, the line joining A with the center O.

To prove $AB=AC$ and $\angle x=\angle z$.

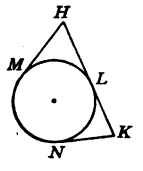
Proof. Draw the radii OB and OC.

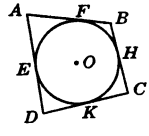
$OB \perp AB$ and $OC \perp AC$. § 288

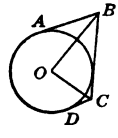
Show that $\triangle ABO=\triangle ACO$.

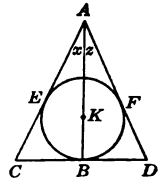
$\therefore AB=AC$ and $\angle x=\angle z$. Why?

EXERCISES









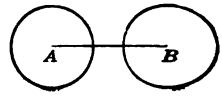
1. Prove that $HK=MH+NK$ in the first figure.

2. The sides of a quadrilateral $ABCD$ are tangent to the circle O. Prove that $AD+BC=AB+DC$.

3. The lines AB, BC, and CD are tangent to the circle O. $AB \parallel DC$. Prove that $\angle O$ is a right angle.

4. The lines AC and AD are tangent to the circle K. A is joined to K and the line is produced to meet the circle at B. CD is tangent to the circle at B. Prove that $\triangle CAD$ is isosceles.

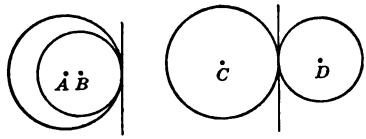
291. Line of Centers. The line joining the centers of two circles is the **line of centers.**



Thus AB is the line of centers of the two circles A and B.

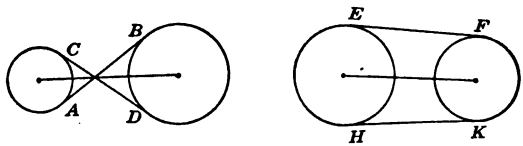
292. Tangent Circles. If two circles are tangent to the same straight line at the same point, they are **tangent circles.**

When both circles are on the same side of the common tangent they are tangent **internally.** When they are on opposite sides of the common tangent, they are tangent **externally.**

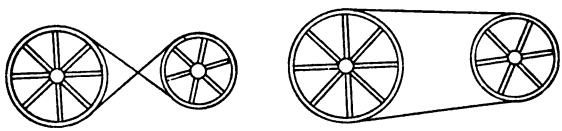


Circles A and B are tangent internally. Circles C and D are tangent externally.

293. Common Tangents. A tangent to two circles which cuts the line of centers between the circles is a **common internal tangent.** A tangent to two circles which does not cut the line of centers between the circles is a **common external tangent.**



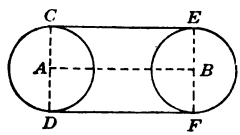
Thus AB and CD are common internal tangents and EF and HK are common external tangents.



The parts of belts between wheels for transmitting circular motion are illustrations of common tangents.

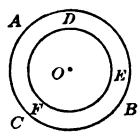
EXERCISE

How can the length of a belt passing around two wheels with equal radii be determined?



294. Concentric Circles. When circles have the same center, they are **concentric circles.**

Thus ABC and DEF are concentric circles, having the same center O.

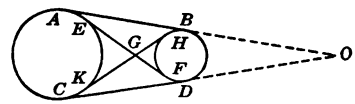


EXERCISES

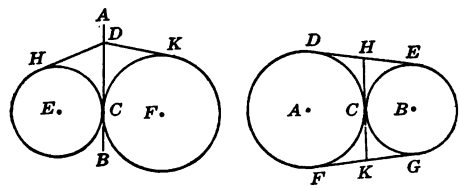
1. Show by drawing how many common tangents can be drawn to two circles when:

(1) The two circles are wholly without each other.
(2) They are tangent externally.
(3) They intersect each other.
(4) They are tangent internally.
(5) One lies wholly within the other.

2. Prove that the common external tangents AB and CD are equal, and that the common internal tangents EF and HK are equal.



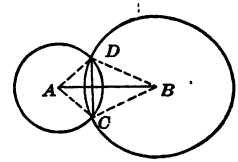
3. The circles E and F are tangent externally. Show that tangents to the circles from any point in the common internal tangent are equal. Let D be any point in the common internal tangent AB. Prove that $DH = DK$.



4. The circles A and B are tangent externally. Prove that the common internal tangent HK bisects the common external tangents and is equal to each of them.

295. Theorem. *The line of centers of two intersecting circles is the perpendicular bisector of their common chord.*

Given AB the line of centers, and DC, the common chord of the two intersecting circles A and B.



To prove that AB is the perpendicular bisector of DC.

Proof. Draw the radii AD, AC, BD, and BC.

$AD = AC$ and $BD = BC$. Why?

Then A and B are each equidistant from D and C.

$\therefore$ AB is the perpendicular bisector of DC. § 207

296. Theorem. *If two circles are tangent to each other, the line of centers passes through the point of contact.*

Given the circles A and B tangent externally in Fig. 1 and tangent internally in Fig. 2. HK is the common tangent.

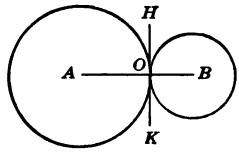


FIG. 1

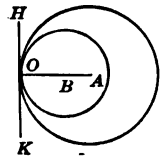


FIG. 2

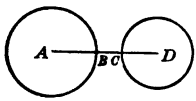
To prove that AB passes through O, the point of contact.

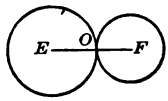
Proof. The radii AO and BO are both perpendicular to HK at O. § 288

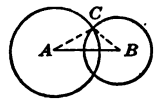
Then AO and BO must be in the same straight line. § 77

$\therefore$ the line of centers must pass through O.

EXERCISES





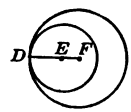


1. When two circles lie wholly without each other, the line of centers is greater than the sum of the radii. § 110

2. When two circles are tangent externally, the line of centers is equal to the sum of the radii. § 109

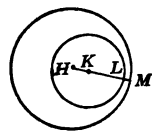
3. When two circles intersect, the line of centers is less than the sum and greater than the difference of the radii. § 182

4. When two circles are tangent internally, the line of centers is equal to the difference of the radii.



$DE+EF=DF$. $\therefore EF=DF-DE$. § 106

5. When one of two circles lies wholly within the other, the line of centers is less than the difference of the radii.



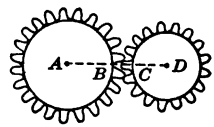
$ML+LK+KH=MH$. § 109

$ML+KH=MH-LK$. § 106

But $KH<ML+KH$. § 110

$\therefore KH<MH-LK$. § 111

6. In a machine, how far apart must the centers of two wheels be placed in order that the cogs may mesh?

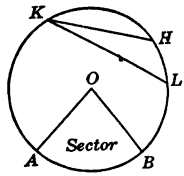


$AD>AB+CD+$ length of cog.

$AD<AB+CD+$ twice length of cog.

ANGLE MEASUREMENT

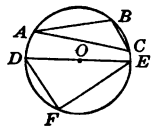
297. Sector. The figure formed by two radii and their intercepted arc is a **sector.**



298. Inscribed Angle. An angle formed by two chords meeting on a circle is an **inscribed angle.**

In the figure, $\angle K$ is an inscribed angle.

299. Angle Inscribed in an Arc. An **angle is inscribed in an arc** if its vertex is on the arc and its sides pass through the ends of the arc.

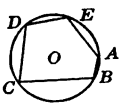


Thus $\angle B$ is inscribed in arc ABC.

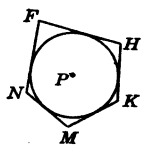
An **angle is inscribed in a semicircle** if its vertex is on the semicircle and its sides pass through the ends of the semicircle.

Thus $\angle F$ is inscribed in the semicircle DFE.

300. A **polygon is inscribed in a circle** if its sides are chords of the circle. It may also be said that the **circle is circumscribed about the polygon.**



A polygon is circumscribed about a circle if its sides are tangent to the circle. It may also be said that the **circle is inscribed in the polygon.**



The polygon $ABCDE$ is an inscribed polygon. The polygon $FHKMN$ is a circumscribed polygon.

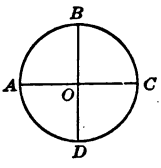
The circle O is a circumscribed circle. The circle P is an inscribed circle.

301. Unit of Measure. In measuring any quantity, some basis or standard is taken which is called the **unit of measure,** as the inch or centimeter.

A **unit of measure** is a standard which is used for measuring quantities of the same kind.

302. Measuring. *To measure a quantity* is to find how many times it contains a unit of measure.

303. Theorem. *The arcs formed by two perpendicular diameters are equal.*



Given AC and BD two perpendicular diameters.

To prove that $\overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}=\overset{\frown}{DA}$.

Proof. $\angle AOB=\angle BOC=\angle COD=\angle DOA$. Why?

$\therefore \overset{\frown}{AB}=\overset{\frown}{BC}=\overset{\frown}{CD}=\overset{\frown}{DA}$. § 272

304. If each of the four angles about O should be divided into 90 equal parts, there would be 360 equal angles about the center O. These angles would intercept 360 equal arcs on the circle, since equal central angles intercept equal arcs in the same circle. § 272

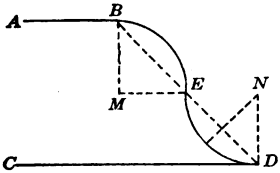
One of the 360 equal angles about a point is taken as the unit of measure for measuring angles and is called an **angle degree.**

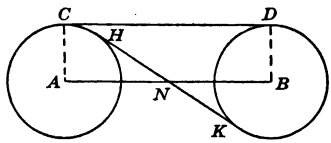
One of the 360 equal arcs, intercepted by the equal angles, is taken as the unit of measure for measuring arcs and is called an **arc degree.**

Then the number of angle degrees in every central angle is the same as the number of arc degrees in its intercepted arc. Because of this fact, the following proposition is true:

305. ***A central angle equals in degrees its intercepted arc.***

EXERCISES

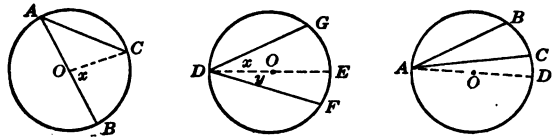




1. The quadrants BE and ED are tangent to each other at E. Show how to draw the curve tangent to AB at B and tangent to CD at D.

2. In the two equal circles, show that the line of centers AB is parallel to the common external tangent CD and bisects the common internal tangent HK.

306. Theorem. ***An inscribed angle equals in degrees one-half its intercepted arc.***



CASE I. **Given** $\angle A$ an inscribed angle having one of its sides a diameter.

To prove $\angle A = {}^\circ \frac{1}{2}\overset{\frown}{BC}$.

Proof. Draw the radius OC.

	$\triangle AOC$ is isosceles.	Why?
Then	$\angle A = \angle C$.	Why?
	$\angle x = \angle A + \angle C$	§ 144
Hence	$\angle A = \frac{1}{2}\angle x$.	Why?
But	$\angle x = {}^\circ \overset{\frown}{BC}$.	§ 305
	$\therefore \angle A = {}^\circ \frac{1}{2}\overset{\frown}{BC}$.	

CASE II. **Given** $\angle D$ an inscribed angle having the center of the circle between its sides.

To prove $\angle D = {}^\circ \frac{1}{2}\overset{\frown}{GF}$.

Proof. Draw the diameter DE.

$\angle y = {}^\circ \frac{1}{2}\overset{\frown}{FE}$ and $\angle x = {}^\circ \frac{1}{2}\overset{\frown}{EG}$. Case I

$\therefore \angle D = {}^\circ \frac{1}{2}\overset{\frown}{FG}$. § 105

CASE III. **Given** $\angle BAC$ an inscribed angle having both of its sides on the same side of the center of the circle.

To prove $\angle CAB = {}^\circ \frac{1}{2}\overset{\frown}{CB}$.

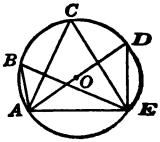
Proof. Draw the diameter AD.

$\angle DAB = {}^\circ \frac{1}{2}\overset{\frown}{DB}$ and $\angle DAC = {}^\circ \frac{1}{2}\overset{\frown}{DC}$. Case I

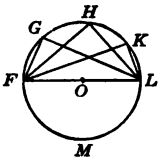
$\therefore \angle CAB = {}^\circ \frac{1}{2}\overset{\frown}{CB}$. § 106

307. Theorem. ***Angles inscribed in the same arc are equal.***

$\angle B = \angle C = \angle D = {}^\circ \frac{1}{2}\overset{\frown}{AE}$. § 306



308. Theorem. ***An angle inscribed in a semicircle is a right angle.***

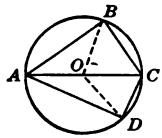


There are 180° in the semicircle FML. Why?

$\angle G = \angle H = \angle K = \circ\ \frac{1}{2}\overset{\frown}{FML}$. § 306

$\therefore$ $\angle G$, $\angle H$, and $\angle K$ are right angles. Why?

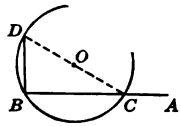
EXERCISES



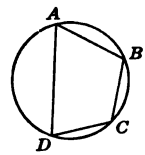
1. Where are the vertices of the right angles of all right triangles having a given hypotenuse?

2. The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

3. Construct a perpendicular to a given line at one of the extremities of the line.



Construct a perpendicular to AB at B. With a convenient point O as a center and the distance from O to B as a radius, describe a circle intersecting AB at C. Draw the diameter CD. Join the points B and D. $BD \perp AB$. Why? How could BD be made perpendicular to AB by means of a tape line?



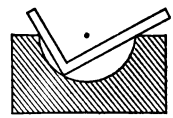
4. The opposite angles of an inscribed quadrilateral are supplementary.

5. A parallelogram inscribed in a circle is a rectangle.

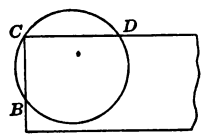
6. In a circle, inscribed angles intercept arcs of 70°, 50°, 82°, 146°, 98°. How many degrees in each angle?

7. In circles, there are inscribed angles of 70°, 20°, 89°, 42°, 27°, 16°. How many degrees in each intercepted arc?

8. A triangle is formed by three chords which subtend arcs of 50°, 200°, and 110°. How many degrees in each angle of the triangle?



9. Pattern makers use a carpenter's square to determine a semicircle. If the vertex touches the curve at all points while the sides rest on the two edges, it is a semicircle. Why? It may be noted that the figure presents an optical illusion.

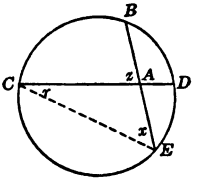


10. Show how a circle may be bisected with a ruler without using the center of the circle. Since angle C is a right angle, BCD is a semicircle.

309. Theorem. *An angle formed by two intersecting chords equals in degrees one-half the sum of the intercepted arcs.*

Given $\angle z$ formed by the two intersecting chords CD and BE.

To prove $\angle z = ^\circ \frac{1}{2}\overset{\frown}{BC} + \frac{1}{2}\overset{\frown}{DE}$ or $\frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{DE})$.



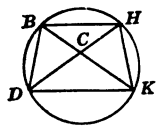
Proof. Draw CE.

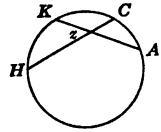
	$\angle z = \angle x + \angle r$.	Why?
But	$\angle x = ^\circ \frac{1}{2}\overset{\frown}{BC}$.	§ 306
And	$\angle r = ^\circ \frac{1}{2}\overset{\frown}{DE}$.	
Then	$\angle x + \angle r = ^\circ \frac{1}{2}\overset{\frown}{BC} + \frac{1}{2}\overset{\frown}{DE}$.	Why?
	$\therefore \angle z = ^\circ \frac{1}{2}\overset{\frown}{BC} + \frac{1}{2}\overset{\frown}{DE}$.	§ 111
Or	$\angle z = ^\circ \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{DE})$.	

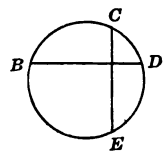
EXERCISES

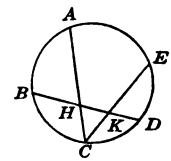
1. In § 309, if $\overset{\frown}{CE} = 135^\circ$ and $\overset{\frown}{BD} = 65^\circ$, how many degrees in $\angle BAD$?

2. $DBHK$ is an isosceles trapezoid. $\overset{\frown}{HK} = 70^\circ$. Find the number of degrees in each of the angles formed by the diagonals.



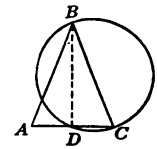






3. Given that $\overset{\frown}{KC} = 60^\circ$, and $\angle z = 50^\circ$. How many degrees in $\overset{\frown}{HA}$?

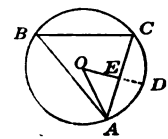
4. If $BD \perp CE$, prove that $\overset{\frown}{BE} + \overset{\frown}{CD} = 180^\circ = \overset{\frown}{BC} + \overset{\frown}{DE}$.



5. The chord BD bisects $\overset{\frown}{AC}$ and $\overset{\frown}{CE}$. Prove that $\triangle CHK$ is isosceles.

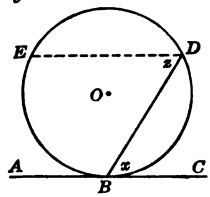
6. A circle described upon one of the equal sides of an isosceles triangle as a diameter, bisects the base of the triangle.

7. If $\angle B$ is inscribed in arc ABC and $OE \perp AC$, prove that $\angle B = \angle O$.



8. The base of an inscribed isosceles triangle subtends an arc of 80°. How many degrees in each angle of the triangle?

310. Theorem. *An angle formed by a tangent and a chord equals in degrees one-half the intercepted arc.*



Given $\angle x$ formed by the tangent AC and the chord BD.

To prove $\angle x = \circ\ \frac{1}{2}\overset{\frown}{BD}$.

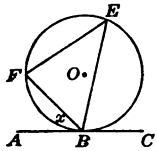
Proof. Draw $ED \parallel AC$.

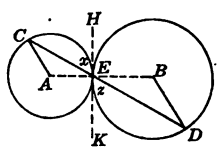
Then	$\overset{\frown}{EB} = \overset{\frown}{BD}$.	Why?
And	$\angle x = \angle z$.	Why?
But	$\angle z = \circ\ \frac{1}{2}\overset{\frown}{EB}$.	§ 306
	$\therefore \angle x = \circ\ \frac{1}{2}\overset{\frown}{BD}$.	§ 111

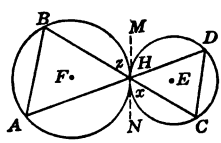
EXERCISES

1. A chord subtends an arc of 70°. A tangent is drawn to the circle at one end of the chord. Find the number of degrees in each angle formed by the tangent and the chord.

2. A chord drawn from the point of contact of a tangent makes with the tangent an angle of 80°. How many degrees has each arc of the circle?



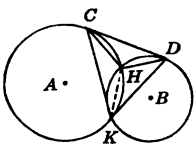


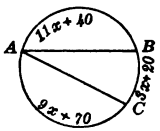


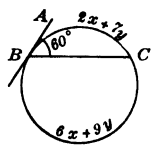
3. If AC is tangent to the circle O at B, prove that $\angle x = \angle E$.

4. The circles A and B are tangent externally. CD is any line drawn through the point of contact. AC and BD are radii of the circles. Prove that $AC \parallel BD$.

5. The circles E and F are tangent externally. AD and BC are any two lines drawn through the point of contact. Prove that $AB \parallel CD$.





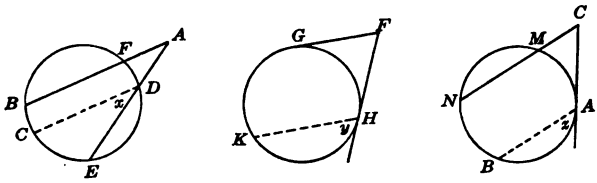


6. Circles A and B intersect at H and K. CD is a common tangent. Prove that $\angle CHD$ is the supplement of $\angle CKD$.

7. Find the number of degrees in $\angle A$.

8. Find the value of x and of y in the last figure.

311. Theorem. *An angle formed by two secants, two tangents, or a tangent and a secant, equals in degrees one-half the difference of the intercepted arcs.*



CASE I. **Given** $\angle A$ formed by two secants, AB and AE.
To prove that $\angle A = ^\circ \frac{1}{2}\overset{\frown}{BE} - \frac{1}{2}\overset{\frown}{FD}$ or $\frac{1}{2}(\overset{\frown}{BE} - \overset{\frown}{FD})$.
Proof. Draw $DC \parallel AB$.

$\overset{\frown}{BC} = \overset{\frown}{FD}$.	Why?
$\angle x = \angle A$.	§ 133
$\angle x = ^\circ \frac{1}{2}\overset{\frown}{CE}$.	Why?
$\overset{\frown}{CE} = \overset{\frown}{BE} - \overset{\frown}{BC}$	
$= \overset{\frown}{BE} - \overset{\frown}{FD}$.	Why?
$\therefore \angle A = ^\circ \frac{1}{2}(\overset{\frown}{BE} - \overset{\frown}{FD})$.	§ 111
Or $\angle A = ^\circ \frac{1}{2}\overset{\frown}{BE} - \frac{1}{2}\overset{\frown}{FD}$.	

CASE II. **Given** $\angle F$ formed by two tangents, FG and FH.
To prove that $\angle F = ^\circ \frac{1}{2}\overset{\frown}{GKH} - \frac{1}{2}\overset{\frown}{GH}$ or $\frac{1}{2}(\overset{\frown}{GKH} - \overset{\frown}{GH})$.
Proof. Draw $HK \parallel FG$.

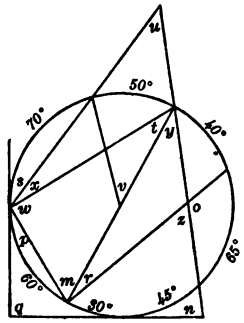
$\overset{\frown}{GK} = \overset{\frown}{GH}$.	Why?
$\angle y = \angle F$.	Why?
$\angle y = ^\circ \frac{1}{2}\overset{\frown}{KH}$.	§ 310
$\overset{\frown}{KH} = \overset{\frown}{GKH} - \overset{\frown}{GK}$	
$= \overset{\frown}{GKH} - \overset{\frown}{GH}$.	Why?
$\therefore \angle F = ^\circ \frac{1}{2}(\overset{\frown}{GKH} - \overset{\frown}{GH})$.	Why?
Or $\angle F = ^\circ \frac{1}{2}\overset{\frown}{GKH} - \frac{1}{2}\overset{\frown}{GH}$.	

CASE III. **Given** $\angle C$ formed by the tangent CA and the secant CN. Proof similar to Case II.

EXERCISES

1. Fill out this table with the number of degrees in each angle.

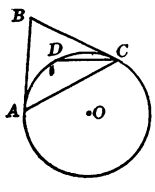
$\angle m = 60°$	$\angle r =$	$\angle w =$
$\angle n =$	$\angle s =$	$\angle x =$
$\angle o =$	$\angle t =$	$\angle y =$
$\angle p =$	$\angle u =$	$\angle z =$
$\angle q =$	$\angle v =$	

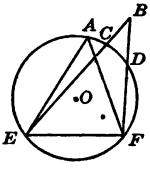


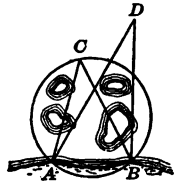
2. An angle of 50° is formed by two tangents. How many degrees in each of the intercepted arcs?

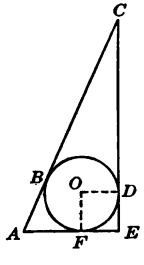
3. Tangents are drawn at each end of a chord which subtends an arc of 90°. What kind of triangle is formed?

4. Tangents are drawn at each end of a chord which subtends an arc of 120°. What kind of triangle is formed?









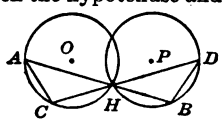
5. BA and BC are tangents to the circle O. D is the middle point of $\overset{\frown}{AC}$. Prove that DC bisects $\angle BCA$.

6. An inscribed angle is greater than an angle formed by two secants which intercept the same arc. Why is $\angle B < \angle A$?

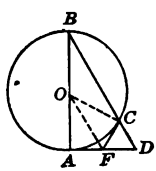
7. Ships avoid rocks and shoals near a coast by what is called the "horizontal danger angle." At A and B are two lighthouses on shore. $\angle C$ is found by observation such that the arc ACB in which it is inscribed includes all the dangerous places. The sailor keeps his vessel so that $\angle D$, formed by the directions to the two lighthouses, is always less than $\angle C$. Will this keep him out of danger? Why?

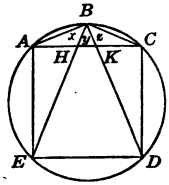
8. A right triangle is circumscribed about a circle. Show that the diameter of the circle is equal to the difference between the hypotenuse and the sum of the other two sides.

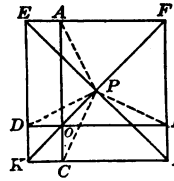
9. The equal circles O and P intersect at H. AB and CD are drawn through the point of intersection. Prove that chord AC = chord BD.



10. Two parallel chords subtend arcs of 70° and 150°. An isosceles trapezoid is formed by joining the ends of the chords. How many degrees in each angle of the trapezoid? Construct the trapezoid in two ways.



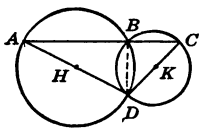


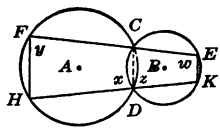


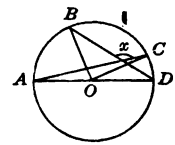
11. In the circle O, AB is a diameter. The hypotenuse of rt.$\triangle ABD$ cuts the circle at C. CF is tangent to the circle. Prove that $AF = FD$.

12. $ACDE$ is a square. B is the middle point of the arc ABC. Prove that $\angle x = \angle y = \angle z$. Also prove that $\triangle HBK$ is isosceles, that $AH = KC$, and that $EHKD$ is an isosceles trapezoid.

13. Through any point O on a diagonal of a square, two lines, AC and DB, are drawn parallel to sides of the square. Prove that the point of intersection P, of the diagonals, is the center of a circle which passes through A, B, C, and D.



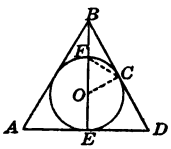


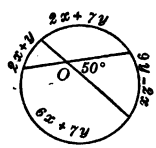


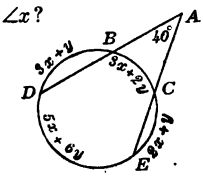
14. Two circles intersect at B and D. From D, the diameters DC and DA are drawn. The point B is joined to A and C by the lines BA and BC. Prove that BA and BC are in the same straight line.

15. Two circles intersect at C and D. Any two straight lines, as FE and HK, are drawn through C and D, terminating in the circles. Prove that the chords HF and KE are parallel.

16. If $\angle BOC$ is a right angle, how many degrees in $\angle x$?





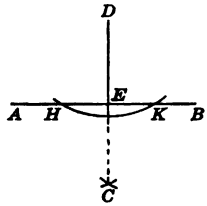


17. The radius of the circle inscribed in an equilateral triangle is equal to one-third the altitude of the triangle.

18. Find the value of x and of y in each of the last two figures.

PROBLEMS OF CONSTRUCTION

312. Problem. ***To construct a perpendicular to a given straight line from a given external point.***



Given the straight line AB and the external point D.

Construction. With D as a center, describe an arc which intersects AB in two points as H and K. With H and K as centers and equal radii, describe arcs which intersect in some point as C. Join D and C.

Proof. D is equidistant from H and K. § 265

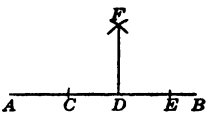
C is also equidistant from H and K. Why?

Then $CD \perp HK$. § 207

$\therefore DE \perp AB$.

Could the point C have been found above the line AB?

313. Problem. ***To construct a perpendicular to a given straight line at a given point in the line.***



Given the straight line AB and the point D in the line.

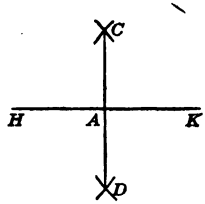
Construction. With D as a center, lay off $DC = DE$. With C and E as centers and equal radii, describe arcs intersecting at F.

Proof. D and F are equidistant from C and E. § 265

Then $FD \perp CE$. Why?

$\therefore FD \perp AB$.

314. Problem. ***To bisect a given straight line.***



Given the straight line HK.

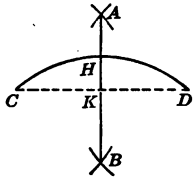
With H and K as centers and equal radii more than one-half of HK; describe arcs intersecting at C and D.

CD is the perpendicular bisector of HK.

$\therefore$ The line HK is bisected at A. § 207

315. Problem. *To bisect a given arc.*

Given the arc CD.

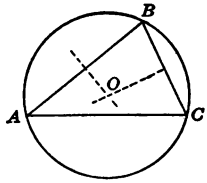


Draw the chord CD. Construct AB, the perpendicular bisector of the chord CD.

Then AB passes through the center of the circle. § 280

$\therefore$ AB bisects CD. § 277

316. Problem. *To circumscribe a circle about a given triangle.*

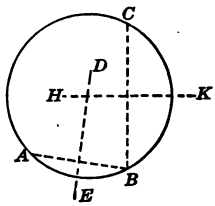


The perpendicular bisectors of the sides of the triangle meet in a point which is equidistant from the three vertices. § 210

Construct the perpendicular bisectors of AB and BC intersecting at O. With O as a center and OB as a radius, describe a circle which passes through A, B, and C, and is circumscribed about the triangle ABC. Why?

317. Problem. ***To construct a circle through three points which are not in the same straight line.***

Given A, B, and C, three points not in a straight line.



Draw AB and BC. Describe the circle passing through the three points A, B, and C in the same way that a circle is circumscribed about $\triangle ABC$ in § 316.

318. Theorem. *Three points not in the same straight line determine a circle.*

Since DE and HK can intersect in only one point, then O is the center of the only circle passing through A, B, and C.

$\therefore$ one circle and only one can pass through three points not in the same straight line, or, three points not in the same straight line determine a circle.

EXERCISES

1. Can a circle pass through three points D, E, and F, which are in the same straight line? Give reason for the answer.

2. Circumscribe a circle about a given right triangle.

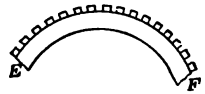
3. Circumscribe a circle about a given obtuse triangle.

4. Find the center of a given circle.

5. Find the radius of a fifty-cent piece.

6. Find the center of the circle of which HK is an arc.

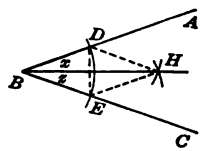
7. In the figure, EF represents part of a broken clock wheel. Find the diameter of the wheel.



319. Problem. ***To bisect a given angle.***

Given the angle ABC.

Draw DH and EH as indicated and show that $\triangle BDH = \triangle BEH$.



Then $\angle x = \angle z$. Why?

$\therefore \angle ABC$ is bisected.

EXERCISES

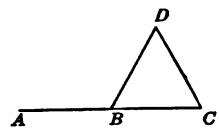
1. Construct an angle of 45°. Of 135°.

2. Construct an angle of 22° 30′. Of 67° 30′.

3. Construct an angle of 60°.

4. Construct an angle of 30°.

5. Trisect a right angle.

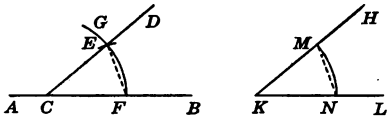


6. Construct an angle of 75°.

7. Show that an angle of 60° may be formed at C with a tape line as follows: The line AC is laid off equal to 20 ft. One person takes the tape at the middle point B. Another person places the end A at B while the person at the middle moves to D, making the halves of the tape, BD and DC straight.

Show that $\triangle BDC$ is an equilateral triangle. $\therefore \angle C = 60°$.

320. Problem. ***From a given point in a given line, to draw a line making an angle equal to a given angle.***



Given the point C in the line AB, and $\angle K$.

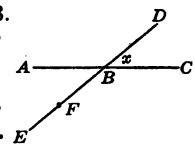
Construction. With K as a center and any convenient radius, describe an arc intersecting the sides of the angle at the points M and N. With C as a center and the same radius, describe the arc GF intersecting AB at F. With F as a center and MN as a radius, describe an arc intersecting GF at E. Draw CD through E. Then $\angle BCD$ is the required angle.

Proof.	Chord EF = chord MN.	§ 265
Then	$\overset{\frown}{EF} = \overset{\frown}{MN}$.	§ 275
	$\therefore \angle BCD = \angle K$.	§ 272

EXERCISES

1. Draw parallel lines through C and D, points in a line AB, by constructing lines perpendicular to AB at C and D by § 313.

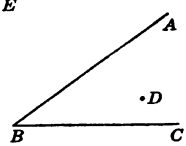
2. Construct a line through point F parallel to AC by making at F an angle equal to $\angle x$ by § 320.



3. Through a point C outside a line AB, construct by two different methods a line parallel to the line AB.

4. Through point D, draw a line which shall make equal angles with the sides of $\angle ABC$.

5. On a given line construct a triangle having its three angles equal respectively to the three angles of a given triangle.



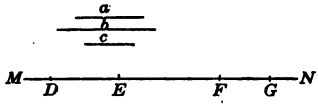
6. Construct a regular octagon having its sides of a given length.

SUGGESTION. Find the interior angle of a regular octagon by § 171 and apply § 320.

7. Suppose P is a point in a street of which AB is the curb. It is required to lay a gas pipe from P perpendicular to the curb. A cord PD is stretched to a point D on the curb. The center C of this cord is held in position while the end P is carried until it just reaches a point E, on the curb. Show that PE is perpendicular to AB.

321. Problem. *Lines may be added, subtracted, and multiplied.*

(1) **To add** the lines a, b, and c.



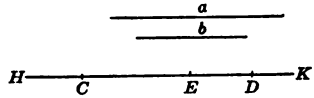
Draw the indefinite line MN. With some point D as a center and a radius equal to a, describe an arc intersecting MN at E. Similarly, with E as a center, make $EF=b$, and, with F as a center, make $FG=c$.

$DG=DE+EF+FG$. Why? $\therefore DG=a+b+c$.

(2) **To subtract** the line b from the line a.

Draw the indefinite line HK.

Make $CD=a$, and $CE=b$.



$CD=CE+ED$. Why?

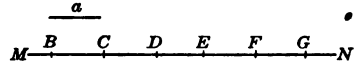
$CD-CE=ED$. Why?

$\therefore a-b=ED$.

(3) **To multiply** the line a by 5.

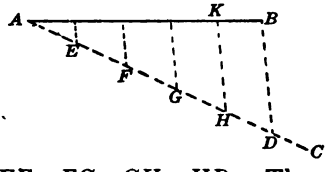
Draw the indefinite line MN. Make BC, CD, DE, EF, and FG each equal to a.

Then $BG=5\,a$.



322. Problem. *A line may be divided into equal parts.*

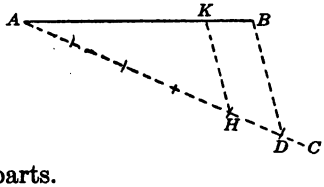
To divide the line AB into any number of equal parts, as 5.



Draw AC, making any convenient angle with AB. With any convenient radius, as AE, describe 5 arcs successively, making $AE=EF=FG=GH=HD$. Then draw DB. Through E, F, G, and H, draw lines parallel to DB.

These lines divide AB into 5 equal parts. § 197

Discussion. In actual work, it is necessary to draw only one line, $HK \parallel DB$. Since KB is one-fifth of AB, by applying KB as a radius successively from A or B, the line AB will be divided into five equal parts.

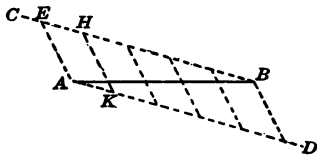


EXERCISES

1. Construct an equilateral triangle whose perimeter is four inches.

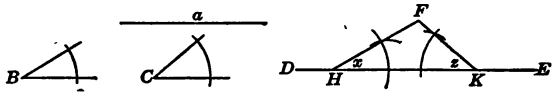
2. Divide a line three inches long into four equal parts by two different methods.

3. Show how a line, as AB, may be divided into equal parts by this figure.



Make BC and AD parallel. Why is $AEHK$ a parallelogram? $\therefore$ AB is divided into equal parts. § 197

323. Problem. *To construct a triangle when two angles and the included side are given.*



Given angles B and C, and a the included side.

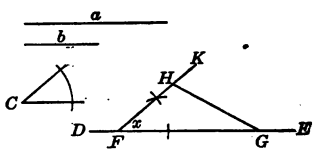
Construction. On the indefinite line DE, lay off $HK = a$. At H construct $\angle x = \angle B$. § 320

At K construct $\angle z = \angle C$. Produce the sides of the angles x and z until they meet at F.

Then $\triangle HFK$ is the required triangle.

324. Problem. *To construct a triangle when two sides and the included angle are given.*

Show how $\triangle FHG$ is constructed with two sides equal respectively to a and b and the included angle equal to $\angle C$.

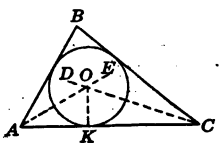


325. Problem. *To inscribe a circle in a given triangle.*

Given the triangle ABC.

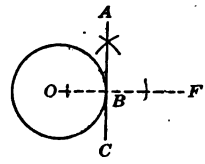
Find the center O of the circle. § 209

Construct $OK \perp AC$. With O as a center and OK as a radius, construct the circle. This circle will touch each of the three sides of the triangle and be inscribed in it, since O is equidistant from the three sides of the triangle.



326. Problem. ***To construct a tangent to a given circle.***

CASE I. ***At a given point on the circle.***



To construct a tangent to the circle at the point B on the circle.

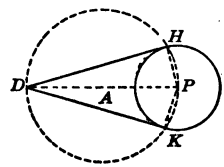
Construction. Draw the radius OB and produce it to some point, as F. Construct $AC \perp OF$ at B.

Why is AC tangent to the circle at B?

CASE II. ***From a point without the circle.***

To construct a tangent to the circle whose center is P, from the point D without the circle.

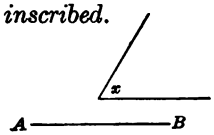
Construction. Join D with the center of the circle P by the line DP. Bisect DP at A. With A as a center and radius AP, describe the circle which intersects the given circle at H and K. Draw DH and DK. Then DH and DK are tangents.



Proof. $\angle H$ and $\angle K$ are right angles. § 308

$\therefore$ DH and DK are tangents to $\odot P$ from point D. Why?

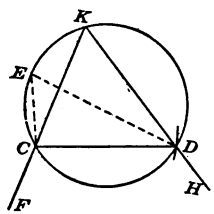
327. Problem. *With a given straight line as a chord, to construct an arc of a circle in which a given angle may be inscribed.*



Given line AB and $\angle x$.

Construct $\angle K = \angle x$. With C, any convenient point on KF, as center, and a radius equal to AB, describe an arc intersecting KH at D.

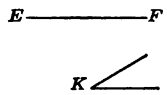
Draw CD and construct the circle passing through C, K, and D. § 317



Then $\overset{\frown}{CKD}$ is the required arc.

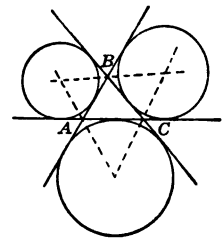
Why is any angle inscribed in $\overset{\frown}{CKD}$, as $\angle E$, equal to $\angle x$? § 307

EXERCISE



Where are the vertices of all triangles having EF as one side and an angle opposite this side equal to $\angle K$?

328. The **escribed circles** of a triangle are the circles, each of which touches one side of the triangle and the other two sides produced.



EXERCISE

Construct the escribed circles of $\triangle ABC$.

SUGGESTION. The centers of the circles are found by bisecting the exterior angles of the triangle.

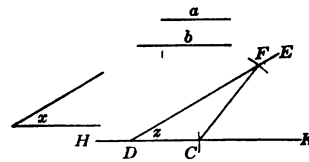
329. Problem. *To construct a triangle when two sides and the angle opposite one of them are given.*

Given a and b the two sides and $\angle x$ opposite b.

If $a<b$ and $\angle x$ is acute.

Construct $\angle z=\angle x$. Lay off $DC=a$ and $CF=b$.

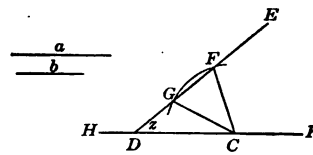
$\triangle DFC$ is the required triangle.



If $a=b$, the triangle is isosceles.

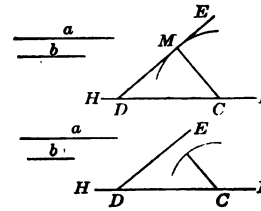
If $a>b$ and $\angle x$ is acute.

Construct $\angle z=\angle x$. Lay off $DC=a$. With C as a center and a radius equal to b, describe an arc. This will intersect DE at two points, G and F, if b is longer than the perpendicular from C to DE. Triangles DGC and DFC both satisfy the conditions and the case is ambiguous.



If b is equal to the distance from C to DE, only one triangle, rt.$\triangle DMC$, is formed.

If b is less than the distance from C to DE, no triangle can be formed.



Construct the triangle with $a<b$, and $\angle x$ a right angle, and also with $a<b$ and $\angle x$ an obtuse angle.

Can a triangle be constructed with $a=b$ or $a>b$ and $\angle x$ a right angle or an obtuse angle?

LOCATION OF POINTS. LOCI.

330. Locus. The location of all points which satisfy a given condition is commonly called a **locus.** This is a Latin word which means *place.* Problems dealing with the location of points are usually designated as **loci problems.**

The *locus of points* or the *locus of a point* may be defined as the path of a point moving according to some fixed law.

For example, the bisector of an angle is the locus of points equidistant from the sides of the angle.

In order to prove that a line or a number of lines is a locus of points, it must be shown:

(1) *That all points in the line or lines satisfy the given condition.*

(2) *That all points which satisfy the given condition are in the line or lines.*

This is illustrated in §§ 199, 200, 205, 206.

To construct a figure which will fulfill two conditions, it is often necessary to find the locus of points satisfying the first requirement and the locus of points satisfying the second requirement. Then the intersection of these two loci satisfies both requirements.

The following problems on loci are to be illustrated by drawing without proof.

EXERCISES

1. Where are all the points one inch distant from a given point? Where are all the points at a given distance from a given point?

2. Where are all the points one-half inch from a circle whose radius is one inch? Where are all the points at a given distance from a given circle?

3. Where are the centers of all circles with a one-inch radius, which pass through a given point?

4. Where are the centers of all circles with one-inch radius tangent to a circle with a radius of two inches?

5. Where are the centers of all circles tangent to two concentric circles whose radii are one inch and two inches respectively?

6. Where are the centers of all circles with radii of one inch, which touch a given straight line?

7. Where are the centers of all circles which are tangent to a given straight line CD at a given point O?

8. Where are the centers of all circles tangent to two parallel lines one inch apart?

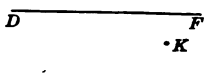
9. Where are all the points equidistant from two intersecting lines? § 200

10. Where are the centers of all circles tangent to two intersecting lines?

11. Where are the centers of all circles which pass through two points E and F which are one inch apart?

12. Find a point equidistant from three given points A, B, and C, which are not in the same straight line.

13. Find a point in a line DF equidistant from two given points H and K.

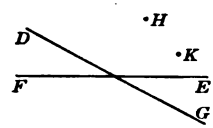


14. Find all the points in a line CD which are one inch distant from a given point B.

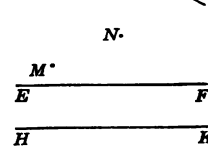
15. Two points A and B are one inch apart. Find all the points which are two inches from A and one and one-half inches from B.

16. Find all the points equidistant from two given points, A and B, and one inch distant from a third point C.

17. Find all the points equidistant from two given points H and K and equidistant from two given intersecting lines, DG and FE.

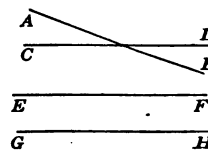


18. Find all the points equidistant from two given points M and N and equidistant from two given parallel lines, EF and HK.



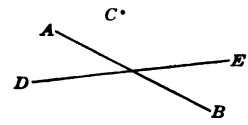
19. Find all points one inch from a given line AC and two inches from a given point B.

20. Find all points equidistant from two given intersecting lines AB and CD and equidistant from two given parallel lines, EF and GH.



21. A house is two miles from a straight railroad track. Locate by a drawing all the points one mile from the railroad and five miles from the house. Use either a centimeter or a half inch to represent a mile.

22. Find all the points equidistant from two given parallel lines EF and GH and at a given distance from a given point A.



23. Find all the points equidistant from two given intersecting lines AB and DE and at a given distance from a given point C.

24. In the side BC of $\triangle ABC$, find the point which is equidistant from the other two sides.

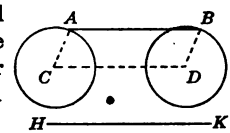
25. Find a point one inch from the vertex of a given angle E, and equidistant from the sides of the angle.

26. Find all the points equidistant from two given intersecting lines AB and CD which are also one inch from the point of intersection O.

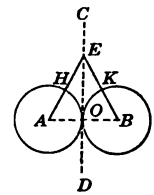
27. Construct a circle with its center in a given line CD, which shall pass through two given points, A and B. Is this always possible?

28. Where are the vertices of all triangles which have a given line AB as a base and an altitude of one inch?

29. A given straight line AB is always parallel to another straight line HK, and moves so that one end B always touches a given circle whose center is D. Where are all the points through which A passes?



30. The equal circles whose centers are A and B are tangent externally at O. Any point E in CD, the common internal tangent, is joined to A and B. Prove that $EH = EK$.



31. Construct a circle with its center at D, tangent to the two equal tangent circles whose centers are A and B, using figure of Exercise 30. If two given circles are tangent externally, where are the centers of all circles which are tangent externally to both of them?

32. What is the locus of the center of a circle which rolls around a square and remains in the same plane as the square?

***33.** Draw and describe loci made by (1) the center of a circle that rolls around another circle; (2) a point in the tire of a wheel rolling on level ground.

CONSTRUCTION OF CIRCLES

Construct a circle with one-inch radius which shall:

1. Pass through two points, one inch apart. Two solutions.

2. Be tangent to two given intersecting lines. Four solutions.

3. Pass through a given point and be tangent to a given line. Two solutions.

4. Be tangent to a given circle and also tangent to a given line. Two solutions.

5. Be tangent to a given circle at a given point.

6. Pass through a given point and be tangent to a given circle.

7. Be tangent to a given line and have its center in another line.

Construct a circle which shall:

8. Pass through a given point and be tangent to two given parallel lines.

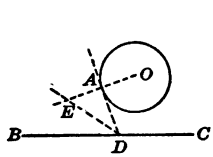
9. Be tangent to three given lines, two of which are parallel.

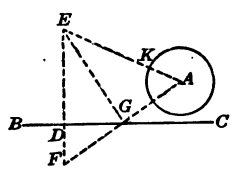
10. Be tangent to the three given lines, no two of which are parallel.

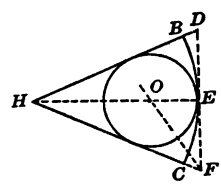
11. Be tangent to a given line at a given point, and have its center in a given line.

12. Be tangent to a given line at a given point and pass through a given point.

13. Be tangent to a given circle at a given point in the circle, and also pass through a given point outside the circle.







14. Be tangent to a given circle O at a given point A in the circle, and also be tangent to a given line BC.

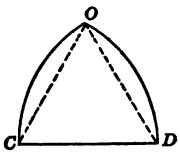
15. Be tangent to a given circle A and also tangent to a given line BC at a given point D in the line.

SUGGESTION. FD is equal to the radius of the circle A.

16. Inscribe a circle in a given sector HBC.

17. Construct three equal circles so that each shall be tangent to a given circle O and also tangent to each of the other two circles.

331. Gothic or Pointed Arch. The figure *COD* shows the form of a Gothic arch. The line *CD* is the **span** of the arch. With *C* and *D* as centers, and *CD* as a radius, two arcs are drawn, intersecting at the point *O*, the **apex** of the arch. Since $\triangle COD$ is an equilateral triangle, the arch *COD* is called an **equilateral Gothic arch.** The Gothic arch may have a span less or greater than the radius of the arcs.



During the Middle Ages, the Gothic arch was introduced into architecture. The attractiveness of many of the great churches and cathedrals is due to this fundamental design. For example, note the beautiful forms in the illustration of Lincoln Cathedral shown on page 9.

EXERCISES

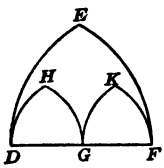


FIG. 1

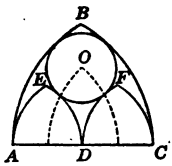


FIG. 2

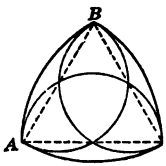


FIG. 3

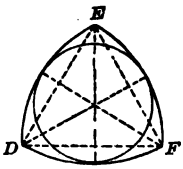


FIG. 4

1. Construct three equilateral Gothic arches, each tangent to the other two, as in Fig. 1.

2. Construct a circle tangent to three equilateral Gothic arches, as in Fig. 2.

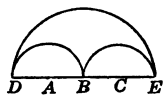


FIG. 5

3. Explain the construction of the window design, with the equilateral triangle *ABC* as its basis. (Fig. 3.)

4. Construct the circle tangent to each of the arcs *ED*, *DF*, and *EF*. (Fig. 4.)

5. Construct three semicircles, each tangent to the other two. (Fig. 5.)

6. Construct and explain the mosaic design in Fig. 6.

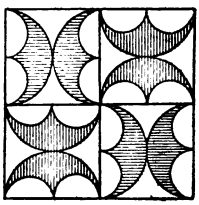


FIG. 6

7. Construct four circles in a square, each circle tangent to two of the other circles. (Figs. 1 and 2.)

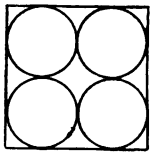


FIG. 1

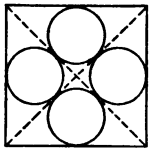


FIG. 2

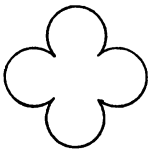


FIG. 3

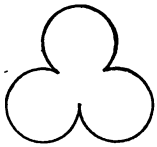


FIG. 4

8. Show the formation of the quatrefoil in Fig. 3 and the trefoil in Fig. 4.

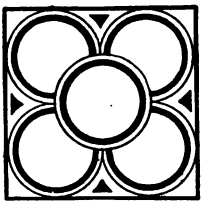


FIG. 5

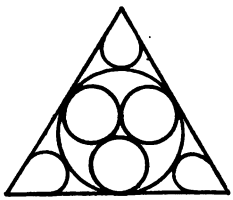


FIG. 6

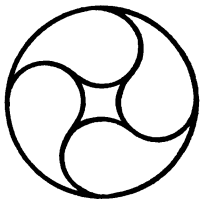


FIG. 7

9. Construct and explain the designs of Figs. 5, 6, and 7.

10. Construct the common external tangents to two circles FK and GL.

SUGGESTION. m is the radius of circle FK. n is the radius of circle GL. $m-n$ is the radius of circle EH. BH is tangent to circle EH. $BL \parallel AK$.

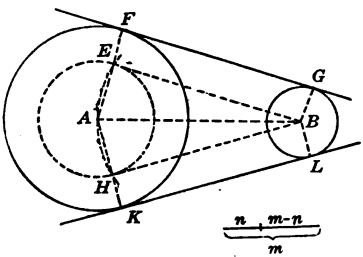


FIG. 8

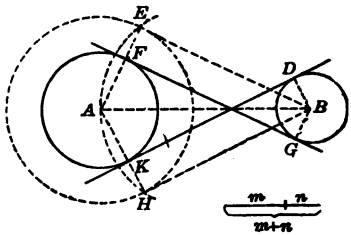


FIG. 9

11. Construct the common internal tangents to two circles FK and DG.

SUGGESTION. m is the radius of circle FK. n is the radius of circle DG. $m+n$ is the radius of circle EH. BE is tangent to circle EH. $BG \parallel EA$.

CONSTRUCTION OF TRIANGLES

Construct a right triangle, having given:

1. The two sides adjacent to the right angle equal to a and b.

2. The hypotenuse equal to c and a side equal to b.

3. A side adjacent to the right angle equal to a and an acute angle equal to $\angle x$.

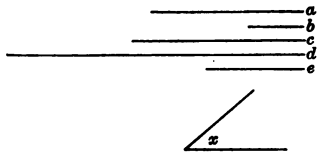


FIG. 1

4. The hypotenuse equal to d and an acute angle equal to $\angle x$.

5. The hypotenuse equal to d and the altitude upon the hypotenuse equal to e.

6. A side adjacent to the right angle equal to c and the altitude upon the hypotenuse equal to e.

7. The median drawn to the hypotenuse equal to a and the altitude upon the hypotenuse equal to e.

8. A side adjacent to the right angle equal to d and the radius of the inscribed circle equal to b.

9. An acute angle equal to $\angle x$ and the radius of the inscribed circle equal to b.

10. The sum of the sides adjacent to the right angle equal to f and an acute angle equal to $\angle z$. (Fig. 2.)

11. The difference between the two sides adjacent to the right angle equal to g and an acute angle equal to $\angle z$. (Fig. 2.)

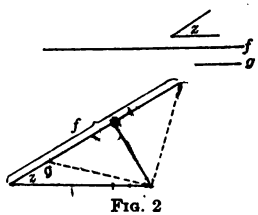


FIG. 2

Construct an equilateral triangle:

12. Having given the altitude.

13. Which shall be inscribed in a given circle.

14. Which shall be circumscribed about a given circle.

Construct an isosceles triangle, having given:

15. The base and the altitude.

16. The base and the angles at the base.

17. The base and the angle at the vertex.

18. The base and the radius of the circumscribed circle.

19. The base and the radius of the inscribed circle.

20. The perimeter equal to e and the altitude equal to f. (Fig. 1.)

21. The perimeter equal to e and the base angles equal to $\angle x$. (Fig. 1.)

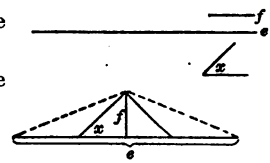


FIG. 1

Construct a triangle, having given:

22. The base, the altitude, and an angle at the base.

23. The base, the altitude, and the angle at the vertex. § 327

24. Two sides and the altitude upon the third side.

25. The base, the median drawn to the base, and the angle at the vertex.

26. Two sides and the median to one of them.

27. One side, an adjacent angle, and the radius of the circumscribed circle.

28. Two angles and the radius of the circumscribed circle.

29. Two angles and the radius of the inscribed circle.

30. The base equal to AB, an angle at the base equal to $\angle x$ and the sum of the other two sides equal to AC. (Fig. 2.)

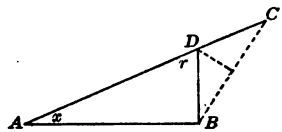


FIG. 2

31. The base equal to HK, an angle at the base equal to $\angle w$, and the difference of the other two sides equal to HE. (Fig. 3.)

32. The base and the altitudes drawn to the other two sides.

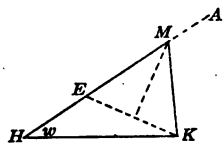


FIG. 3

33. The base, the altitude upon one of the other sides, and the angle at the vertex. § 327

34. An angle, the bisector of this angle, and the altitude drawn from the vertex of this angle.

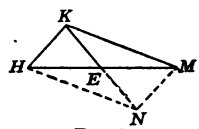


FIG. 4

35. One side, the altitude and median drawn from the same extremity of this side.

36. One side equal to HK, a second side equal to KM, and the median to the third side equal to KE. (Fig. 4.)

37. The three medians equal respectively to DH, CK, and EF. (Fig. 5.) § 211

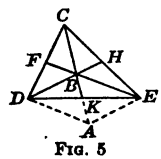


FIG. 5

CONSTRUCTION OF QUADRILATERALS

Construct a square, having given:

1. Its side. **2.** Its diagonal.

3. The sum of the diagonal and one side.

Construct a rectangle, having given:

4. Two adjacent sides. **5.** One side and a diagonal.

6. One side and the angle between the diagonals.

7. The diagonals and the angle between them.

Construct a rhombus, having given:

8. The diagonals. **9.** One side and one angle.

10. One diagonal and one angle.

Construct a parallelogram, having given:

11. One side and the diagonals.

12. The diagonals and the angle between them.

13. Two adjacent sides and one diagonal.

14. Two sides and one angle.

15. One side, the altitude upon that side, and one angle.

Construct an isosceles trapezoid, having given:

16. The bases and one angle. **17.** Two adjacent sides and one angle.

18. The bases and diagonals. **19.** The bases and the altitude.

20. The bases and the non-parallel sides.

Construct a trapezoid, having given:

21. The bases equal to c and d, and the non-parallel sides equal to a and b. (Fig. 1.) Ex. 29, page 135

22. The bases equal to a and b, and the diagonals equal to c and d.

23. The bases equal to c and d, a third side equal to a, and the altitude equal to b.

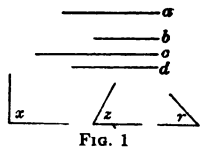


FIG. 1

24. The bases equal to a and b, one diagonal equal to d, and one angle between the diagonals equal to $\angle x$. (Fig. 2.)

25. The bases equal to a and b and the angles at the extremities of a equal to $\angle z$ and $\angle r$.

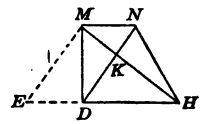


FIG. 2

Construct a quadrilateral, having given:

26. The four sides equal to a, b, c, and d, and an angle equal to 60°.

27. Three sides equal to a, b, and c; the angle formed by a and b equal to 105°; and the angle formed by b and c equal to 60°.

MISCELLANEOUS EXERCISES

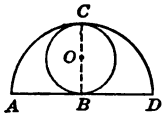


FIG. 1

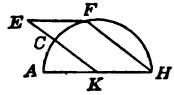


FIG. 2

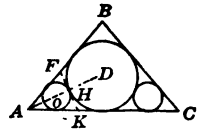


FIG. 3

1. Construct a circle tangent to a given semicircle ACD, and also tangent to the diameter AD. (Fig. 1.)

2. If $EFHK$ is a parallelogram, and $AK = KH$, prove that EK bisects $\overset{\frown}{AF}$. (Fig. 2.)

3. Show how to construct circle O tangent to circle D, and also tangent to AB and AC. AD bisects $\angle A$. (Fig. 3.)

4. Tracks at street intersections sometimes pass around arcs of circles tangent to the straight track. Show how the center of the arcs is found. (Fig. 4.)

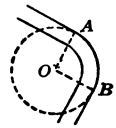


FIG. 4

5. Describe circles about the vertices of a triangle so that each shall be tangent to the other two. (Fig. 5.)

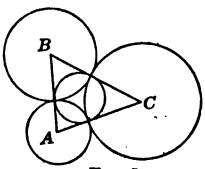


FIG. 5

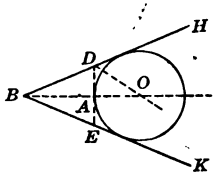


FIG. 6

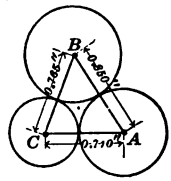


FIG. 7

6. Construct a circle which shall be tangent to the sides of a given angle, HBK, and pass through a given point A on the bisector of the angle. (Fig. 6.) SUGGESTION. $DE \perp AB$. DO bisects $\angle HDA$.

7. In locating and drilling holes, a machinist must find the diameters of three circular disks that, placed tangent to each other as shown in Fig. 7, will have their centers 0.765 in., 0.710 in., and 0.850 in. apart. Find the diameters of the disks.

Show that however placed each tangent to the other two the distances between centers of disks will be the same.

8. A given line AC, one inch long, moves so that its ends always touch two given perpendicular lines EF and BD, making right triangles. Where are all points through which the middle point K of the hypotenuse passes? (Fig. 8.)

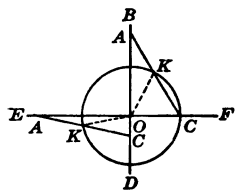
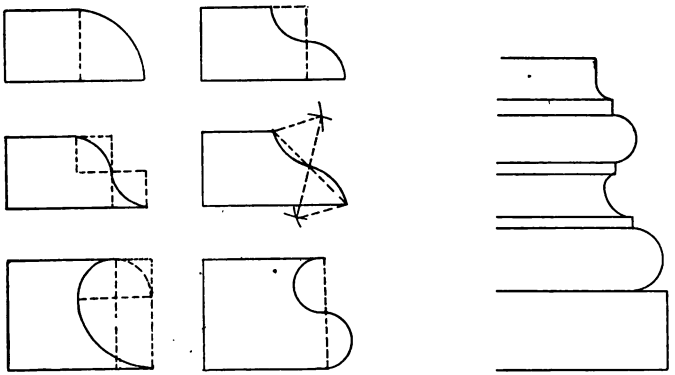
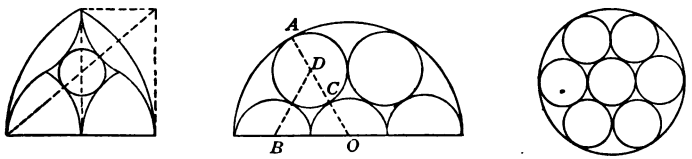


FIG. 8

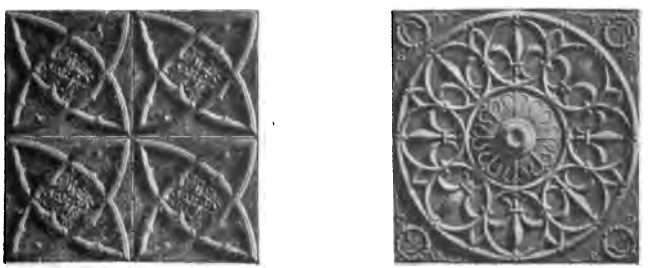
9. Moldings are very important architectural features. They are used to improve the appearance of the angles and projections in panels, cornices, arches, doors, and windows of buildings. The shape of moldings varies according to the style of architecture. Roman moldings are formed from arcs of circles and straight lines. Some of these forms are shown in the figures. Draw them on a larger scale and explain.



10. Construct the following window designs. $\triangle BDO$ is equilateral.



11. Show how the following steel ceiling designs are formed.



12. In a given equilateral triangle ABC (Fig. 1), construct three circles, each tangent to two sides of the triangle, and to each of the other two circles.

SUGGESTION. BF bisects $\angle B$, and EE bisects $\angle BFA$. EK is the radius of the three circles required.

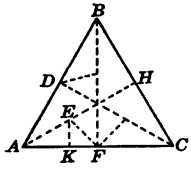


FIG. 1

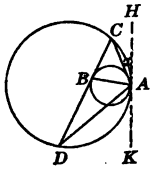


FIG. 2

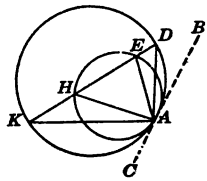


FIG. 3

13. Prove that AB bisects $\angle CAD$. Does $\angle x = \angle D$? Also does $\angle CBA = \angle BAH$? (Fig. 2.)

14. Prove that $\angle DAE = \angle HAK$. (Fig. 3.)

QUESTIONS

1. State theorems concerning angles formed by radii; chords; secants; tangents; chords and tangents.

2. State as many facts as possible about the perpendicular bisector of a chord.

3. What facts can be stated about the angles of an inscribed quadrilateral? Of a circumscribed quadrilateral?

4. Give as many ways as you can for drawing equal angles. Parallel lines.

5. Name as many sets of conditions as you can that will determine a circle.

6. Given two tangents to a circle and two radii drawn to the points of tangency. State the relations of the angles between the tangents and between the radii.

7. State as many loci problems as you can that involve chords in a circle. That involve tangent circles.

8. Give several ways for bisecting an arc. Can an arc be trisected by geometric methods? What are geometric methods? Can a straight line be trisected by geometric methods?

9. Show how to construct a right triangle that has a given hypotenuse.

10. Name several sets of given parts that will determine a triangle uniquely.

11. Make a list of the most useful facts of the present chapter.

CHAPTER III

AREAS OF POLYGONS

MEASURING

332. Definitions. To measure a quantity of any kind is to find how many times it contains a *unit of measure.*

The number of times a quantity contains the unit of measure is called the **numerical measure** of that quantity.

The unit of measure may be any convenient quantity of the same kind as that which is to be measured. Certain units are determined by law and are called **standard units.** In the United States the **meter** is the standard unit of length from which all other units of length are determined.

A convenient unit for measuring the length of the school room is a foot or a meter; for measuring the length of a book, the inch or the centimeter is convenient. The machinist often uses a thousandth of an inch as a unit.

333. *Every geometric magnitude has a numerical measure.*

The numerical measure may be an integer, a fraction, or an irrational number. For example, a segment of a line may have a length of 5 inches, $5\frac{3}{4}$ inches, or $\sqrt{2}$ inches.

334. Theoretically, the *numerical measure*, or, simply the *measure* of a quantity, means its *exact* measurement. Practically, a quantity can be measured only *approximately.*

Thus, one cannot say that a board is exactly 10 feet long. It may be a fraction of an inch more or less than 10 feet. The diameter of a steel rod may be given as 0.457 inch, but it may differ from this by a few ten-thousandths of an inch.

335. Commensurable and Incommensurable Quantities. If two quantities are such that a common unit of measure is contained in each of them an integral number of times, they are said to be **commensurable.** If two quantities are such that

no unit can be found that is contained in each an integral number of times, they are said to be **incommensurable.**

Thus, two lines respectively 7 in. and 9 in. long are commensurable, the common unit of measure being 1 in. It is easy to see that any fractional part of this unit, also, could be used as a common unit of measure. Two lines that are $3\frac{7}{16}$ in. and $4\frac{7}{10}$ in. long respectively, have as a common unit of measure $\frac{1}{80}$ in. Their numerical measures are respectively the quotients found by dividing $3\frac{7}{16}$ and $4\frac{7}{10}$ by $\frac{1}{80}$, or 275 and 376.

336. Two lines 1 in. long and $\sqrt{2}$ in. long respectively, are incommensurable, because a unit of measure that will be contained an integral number of times in 1 in. must be some definite fraction of an inch in length, as $\frac{1}{1000}$ in., and no fractional part of an inch exists that will be contained an integral number of times in $\sqrt{2}$ in. However, a unit can be found such that if it is taken an integral number of times it will approximate $\sqrt{2}$ in. as closely as we please. Thus, since $\sqrt{2}=1.4142\cdots$, if .0001 in. is taken as a unit, 14142 times this unit equals $\sqrt{2}$ within one ten-thousandth of an inch.

NOTE. It should be noticed that any two rational numbers are commensurable, while a rational number and an irrational number are incommensurable.

EXERCISES

***1.** State which of the following pairs of numbers are commensurable, and state the common unit of measure and their numerical measures when commensurable.

(1) 25 in., 72 in.
(2) 17 ft., 44 in.
(3) $27\frac{2}{3}$ yd., 16 ft.
(4) 3 m., 14 cm.
(5) $.333\cdots$ in., 4 in.
(6) $\sqrt{3}$ in., 3 in.
(7) $\sqrt{5}$ ft., $\sqrt{10}$ ft.
(8) 1 yd., 1 m.

(By law of Congress 1 m. = 39.37 in.)

***2.** If two quantities have a common unit of measure how many common units of measure have they? Have they a greatest common unit of measure? Have they a least common unit of measure?

337. Ratio. A method of comparing two quantities of the same kind is to find the ratio of one to the other.

The **ratio** of two quantities of the same kind is the quotient of their numerical measures when the same unit of measure is applied to both.

The quotient is often spoken of as the **value of the ratio.**

Thus, if a and b are the numerical measures of two quantities, their ratio is expressed as $\frac{a}{b}$, or $a : b$.

338. The two numbers used in expressing a ratio are called the **terms** of the ratio. The first term of the ratio is called the **antecedent** or **numerator,** and the second term is called the **consequent** or **denominator.**

339. Since the numerical measures of quantities are numbers, as also are their ratios, they are subject to the same laws of operation as other arithmetical or algebraic numbers. Since ratios are fractions, their properties are those of fractions. Therefore:

340. (1) *Multiplying or dividing both terms of a ratio by the same number does not change the value of the ratio.*

(2) *Multiplying the antecedent, or dividing the consequent, of a ratio multiplies the value of the ratio.*

(3) *Dividing the antecedent or multiplying the consequent of a ratio divides the value of the ratio.*

341. Proportion. If two ratios are equal, as $\frac{a}{b}=\frac{c}{d}$, the numbers a, b, c, and d are said to be in proportion, and the statement of equality is called a **proportion.**

342. A proportion may be written in any one of the following forms: $\frac{a}{b}=\frac{c}{d}$, $a : b = c : d$, $a : b :: c : d$. Each is read "a is to b as c is to d," or "the ratio of a to b equals the ratio of c to d."

The expressions "in the same ratio," "in the same proportion," "proportionally," and "pro rata," all have practically the same meaning.

EXERCISES

1. Find the value of each of the following ratios: $8 : 2$; $3 : 12$; $9\frac{1}{2} : 3\frac{1}{6}$; $2\frac{1}{2} : 9$; 44 cm. : 3 cm.; $2\frac{1}{2}$ in. : 17 in.; $9\frac{4}{5} : 16$; $2.5 : .04$; $a^2-b^2 : a-b$; $a^2+b^2 : a+b$.

2. A room is 20 ft. long and 16 ft wide. Find the ratio of its length to its width. Of its width to its length.

3. One line has a length of 8 in. and a second a length of 20 in. What is the ratio of their lengths? What part is the first of the second? What per cent? How many times as long as the first is the second? What difference is there in the ideas involved in these questions?

4. Divide 70 ft. into two parts in the ratio of $3 : 7$. In the ratio of $2 : 5$.

5. Divide a line 80 in. long into four parts in the ratio of $1 : 2 : 3 : 4$.

6. A man traveled 250 miles, partly by rail and partly by boat. What distance did he travel by each if the distance by rail is to the distance by boat as 3 is to 2?

7. Two supplementary angles are in the ratio of $3 : 5$. Find the angles.

8. The complements of two angles are in the ratio of $5 : 7$. Find the angles if their supplements are in the ratio of $7 : 8$.

9. The sides of a triangle are in the ratio of $5 : 6 : 7$, and the perimeter is 72 in. Find the length of each side.

10. Find the three angles of a triangle that are in the ratio of $7 : 8 : 3$.

11. A triangle has a perimeter of 27 in., and its sides are in the ratio of $6 : 7 : 8$. Find the lengths of the sides of the triangle.

Ans. $7\frac{5}{7}$ in.; 9 in.; $10\frac{2}{7}$ in.

12. The sides of a rectangle are in the ratio of $3 : 7$, and its perimeter is 150 in. Find the dimensions of the rectangle. *Ans.* $22\frac{1}{2}$ in. by $52\frac{1}{2}$ in.

13. The sides of a quadrilateral are in the ratio of $2 : 5 : 8 : 7$, and its perimeter is 77 in. Find the lengths of the sides of the quadrilateral.

14. The angles of a quadrilateral are in the ratio of $2 : 7 : 3 : 6$; find their values.

15. The perimeter of a hexagon is $12\frac{1}{4}$ ft., and its sides are in the ratio of $1 : 2 : 3 : 4 : 5 : 6$. Find the lengths of its sides in inches.

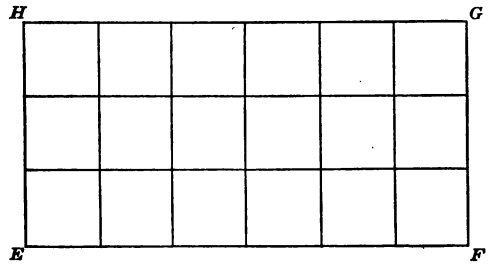
THE RECTANGLE

343. The **numerical measure of a surface** with respect to a unit of area is the number of times the surface contains the unit of area.

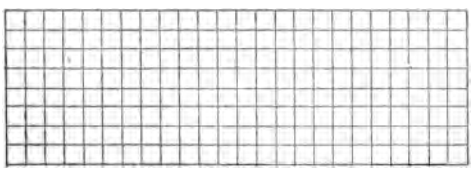
By the *area of a surface* is meant its numerical measure with respect to some unit of measure.

The most convenient units of area are squares one linear unit on a side. Such are the square meter, square foot, square centimeter, and square inch. An acre is a unit of area that is not one linear unit on a side.

344. The figure *EFGH* is a rectangle 3 cm. wide and 6 cm. long. This rectangle can evidently be divided into 3 strips of 6 sq. cm. each, or into 6 strips of 3 sq. cm. each, and therefore its *area* in square centimeters is 3×6 sq. cm., or 6×3 sq. cm., that is, it contains the unit of area 18 times.



In like manner, if the length of a rectangle is 2.4 in. and its breadth is .8 in., one-tenth of an inch may be taken as the linear unit and the rectangle can be divided into 8 strips, each

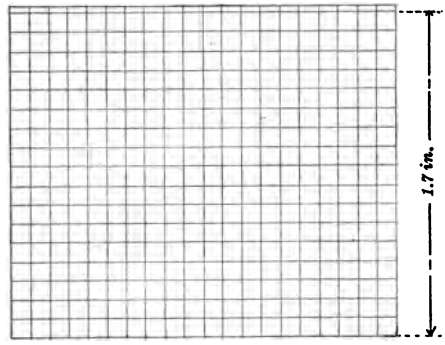


2.4 in. by .8 in.

□
Unit

containing 24 squares .1 in. on an edge. The *area* is then 8×24 of these small squares of which 10×10 make a square inch.

Further, if the length of a rectangle is 2 in. and its width is $\sqrt{3}$ in., the rectangle cannot be exactly divided into small squares no matter how small a unit is taken. Since $\sqrt{3} = 1.73205\cdots$, if .1 in. is taken as a linear unit, the rectangle will contain 17 strips having 20 squares in each, besides a



2 in. by $\sqrt{3}$ in.

remainder strip less than .1 in. wide, and 2 in. long. If .01 in. is taken as a linear unit, the rectangle will contain 173 strips having 200 squares in each, besides a remainder strip less than .01 in. wide and 2 in. long.

If, then, .1 in., .01 in., .001 in., etc. are successively taken as linear units, areas will be found which differ from the true area by remainder strips less than the successive values:

$2 \times .1 = .2$ of a square inch.
$2 \times .01 = .02$ of a square inch.
$2 \times .001 = .002$ of a square inch.
. . . . . .

345. It is evident then that:

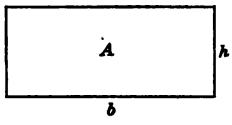
(1) *If the length and the width of a rectangle have a common unit of measure, the area of the rectangle can be expressed exactly in square units.*

(2) *If the length and the width of a rectangle have no common unit of measure, that is, are incommensurable, the area of the rectangle can be approximated to any desired degree of accuracy by taking a sufficiently small unit of measure.*

From the foregoing considerations the truth of the following statement may be accepted:

346. ***The area of a rectangle is equal to the product of its base and altitude.***

If A, b, and h are the numerical measures of the area, base, and altitude respectively of any rectangle, then the theorem is stated in the formula, $A = bh$.



347. Theorem. *The area of a square equals the square of its side.*

348. Theorem. *The areas of two rectangles are to each other as the products of their bases and altitudes.* $\frac{A}{A'} = \frac{bh}{b'h'}$.

349. Theorem. *The areas of two rectangles having equal bases are to each other as their altitudes.*

350. Theorem. *The areas of two rectangles having equal altitudes are to each other as their bases.*

351. Theorem. *Two rectangles having equal bases and equal altitudes are equal in area.*

The preceding five theorems may be considered as corollaries of § 346. (See § 243.)

352. Definition. When two plane geometric figures are equal in the sense that they have the same area they are said to be **equal in area,** or **equivalent.**

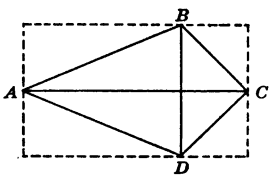
353. For "equal in area," or "equivalent" the symbol = is commonly used. The word "equal" is often used in the sense of equivalent.

In propositions that relate to areas, the words, "rectangle," "parallelogram," "triangle," are often used in the sense of "area of rectangle," "area of parallelogram," "area of triangle."

EXERCISES

1. Prove that the area of a right triangle equals one-half the product of its base by its altitude.

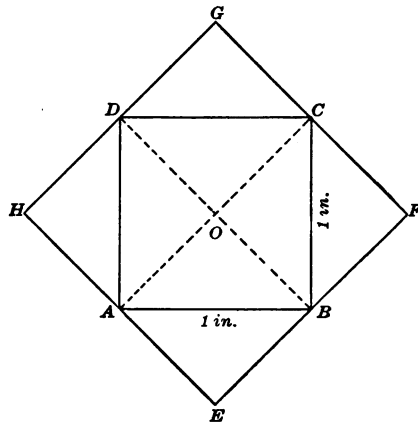
2. Prove that the area of the kite $ABCD$ equals one-half the product of its diagonals, that is, the area equals $\frac{1}{2}(AC \times BD)$.



*3. What common unit can be taken for the length and width in each of the following rectangles: 0.26 in. by 7.2 in.; 1.2 in. by 1.3 ft.; 1.275 in. by 3.2964 in.; 3.265 mi. by 45 rd.; $\frac{3}{16}$ in. by $\frac{2}{5}$ ft.; 3 cm. by 2 in.; 54 ft. by 6 m.; .333 ··· in. by .2424 ··· in.?

*4. Find the area of each of the rectangles in Exercise 3, and state the unit of area.

*5. In the figure, $ABCD$ is a square 1 in. on a side. What is its area? What is the area of the square $EFGH$? Is EF equal to $\sqrt{2}$ in.? Expressed decimally, $\sqrt{2}=1.4142\cdots$. Also, $\sqrt{2}\times\sqrt{2}=2$. By carrying out the following, show that $1.4142\cdots\times1.4142\cdots$ approximates more and more nearly to 2 the more decimal places there are taken in $1.4142\cdots$:



$1\times1=1$

$1.4\times1.4=1.96$

$1.41\times1.41=$

$1.414\times1.414=$

$1.4142\times1.4142=$

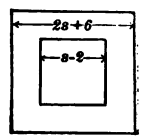
$1.41421\times1.41421=$

By how much does each differ from 2? Express the values of the linear and square units in each case.

6. In measuring a rectangle that is 3.25 in. by 2.46 in., a student makes the measurement of the length 3.26 in. and the width 2.48 in. What is his per cent of error in each measurement? What is the per cent of error in the area computed from his measurements?

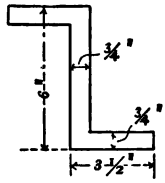
7. In measuring a rectangle the length is measured 2% too large and the width 3% too small. Find the per cent of error in the area computed from these measurements. *Ans.* 1%, nearly, too small.

8. A hollow square has dimensions as shown in the figure. Find its area.



9. How many paving blocks each 4 in. by 4 in. by 10 in., placed on their sides, will it take to pave an alley 600 ft. long and 12 ft. 6 in. wide?

10. Find the area of the cross section of a Z-bar, with dimensions as shown in the figure.

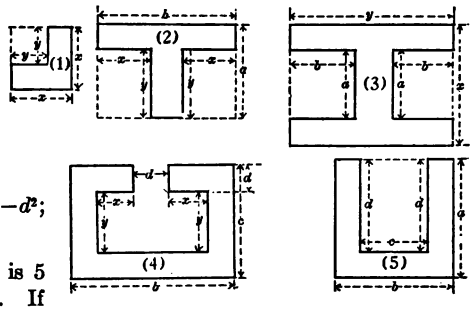


11. A rectangular garden 56 ft. long and 40 ft. wide has a walk 6 ft. wide around it. Find the cost of cementing the walk at 15 cents a square foot. *Ans.* $194.40.

12. The perimeter of a rectangle is 96 ft. and the length is three times the width. Find the area.

13. The area of a rectangle is 144 sq. ft. and its perimeter is 50 ft. Find its dimensions.

14. Find the area of each form given in the figure.



Ans. (1) x^2-y^2;

(2) $ab-2xy$;

(3) $xy-2ab$;

(4) $bc-y(2x+d)-d^2$;

(5) $ab-cd$.

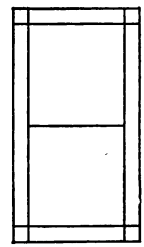
15. A rectangular field is 5 rd. longer than it is wide. If it were 2 rd. wider and 3 rd. shorter it would contain 4 sq. rd. less. Find the dimensions of the field. *Ans.* 8 rd. by 13 rd.

16. A rectangle is 8 in. longer than it is wide. Find the dimensions if the area is 240 sq. in. *Ans.* 12 in. by 20 in.

17. Find the dimensions of a rectangle that is 4 rd. longer than it is wide, if, when the length is increased by 6 rd. and the width by 4 rd., the area is doubled. *Ans.* 10 rd. by 14 rd.

18. A rectangle has an area of 120 sq. in. and its altitude is 2 in. longer than its base. Find the altitude and base.

19. A window has dimensions of 4 ft. and 7 ft. It is to be divided into six squares and four rectangles as shown in the figure. Find the dimensions of the parts.



20. Find the ratio of the areas of two rectangles whose bases and altitudes are respectively 5 in. and 7 in., and 9 in. and 13 in.

***21.** Draw the following to scale, using $\frac{1}{16}$ in. to the rod, then find the area in acres. Start at a point A, go east 20 rd. to B, north 10 rd. to C, east 10 rd. to D, north 40 rd. to E, west 40 rd. to F, south 20 rd. to G, east 10 rd. to H, and south to A.

*22. Draw the plan of the basement of a building to scale as follows: Start at A, go north 12 ft. to B, east 5 ft. to C, north 8 ft. to D, west 5 ft. to E, north 12 ft. to F, west 8 ft. to G, north 10 ft. to H, west 14 ft. to I, south to J, and east to A. Find the cost of cementing it at 15 cents per square foot. *Ans.* $132.60.

23. Prove that the area of a rhombus is half the product of the lengths of its diagonals.

24. The area of a rhombus is 85 sq. ft., and the ratio of the diagonals is as 4 : 7. Find the lengths of the diagonals.

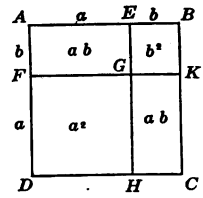
Ans. 9.856 ft. and 17.248 ft.

25. Construct a rectangle whose area is three times that of a given rectangle.

26. Construct a rectangle whose area is to the area of a given rectangle as 3 : 5.

27. The square formed on the sum of two lines is equivalent to the sum of the squares formed on each, plus twice the rectangle formed by the two lines. Or, using a and b as the numerical measures of the lines, the theorem is stated in the formula,

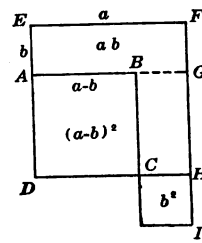
$$(a+b)^2=a^2+2ab+b^2.$$



Given AB the sum of the two lines with numerical measures a and b respectively, and the squares and rectangles formed as shown in the figure; give the construction and proof in detail.

28. The square formed on the difference of two lines is equivalent to the sum of the squares formed on each line, minus twice the rectangle formed by the two lines. Or, using a and b as the numerical measures of the lines, the theorem is stated in the formula,

$$(a-b)^2=a^2-2ab+b^2.$$



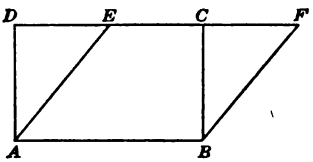
Given $AB=a-b$, where a and b are respectively the numerical measures of the two lines, and the squares and rectangles formed as shown in the figure; give the construction and proof in detail.

29. Draw figures to represent each of the following formulas and explain the constructions:

(1) $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$
(2) $(a+b)(a+c)=a^2+ac+ab+bc.$
(3) $(a+b)(c+d)=ac+ad+bc+bd.$
(4) $(a+b)(a-b)=a^2-b^2.$

THE PARALLELOGRAM

354. Theorem. ***Any parallelogram is equivalent to a rectangle that has its base and altitude equal respectively to the base and altitude of the parallelogram.***



Given $\square ABCD$ and $\square ABFE$ on the same base AB and having the same altitude AD.

To prove $\square ABFE = \square ABCD$.

Proof.

	$\angle AED = \angle BFC$.	Why?
	$AE = BF$.	Why?
Then	rt.$\triangle AED$ = rt.$\triangle BFC$.	Why?
	$ABFD - \triangle AED = \square ABFE$.	
	$ABFD - \triangle BFC = \square ABCD$.	
	$\therefore \square ABFE = \square ABCD$.	Why?

355. Theorem. ***The area of a parallelogram is equal to the product of its base and altitude.*** As a formula, $A = bh$.

This follows from §§ 346 and 354.

356. Theorem. *The altitude of a parallelogram is equal to its area divided by its base.* As a formula, $h = \frac{A}{b}$.

357. Theorem. *The base of a parallelogram is equal to its area divided by its altitude.* As a formula, $b = \frac{A}{h}$.

358. Theorem. *Parallelograms having equal bases and equal altitudes are equivalent.*

359. Theorem. *Two parallelograms having equal bases are to each other as their altitudes.*

360. Theorem. *Two parallelograms having equal altitudes are to each other as their bases.*

361. Theorem. *Any two parallelograms are to each other as the products of their bases and their altitudes.*

EXERCISES

1. Show that the three parallelograms in Fig. 1 are equivalent.

2. Will a yard of velvet "cut on the bias" contain as much material as a yard cut at right angles to the edge? Why?

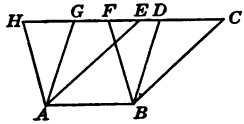


FIG. 1

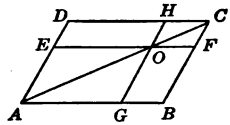


FIG. 2

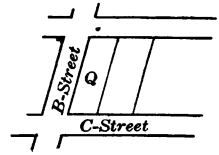


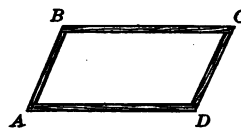
FIG. 3

3. Through a point in a diagonal of a parallelogram, lines are drawn parallel to the sides, thus forming four parallelograms. Prove that $\square EDHO = \square GOFB$. (Fig. 2.)

4. In Fig. 3, *B*-Street and *C*-Street do not cross at right angles. The lot Q extends 150 ft. along *B*-Street and 30 ft. along *C*-Street. Is it possible to find its area from these measurements? Why?

5. What measurements are necessary, in order that the area of a parallelogram can be found?

6. The framework *ABCD* is jointed at its vertices. The opposite sides are equal. If $AB = 8$ in. and $BC = 14$ in., what is the maximum area of the figure that can be formed?



7. Find the area of a parallelogram having a base of 27.3 in. and an altitude of 15.4 in. One having a base of $27\frac{3}{7}$ in. and an altitude of $14\frac{2}{3}$ in.

8. Find the altitude of a parallelogram having an area of 73.5 sq. in. and a base of 12.5 in.

9. Find the base of a parallelogram having an area of $80\frac{3}{14}$ sq. ft. and an altitude of $7\frac{2}{7}$ ft.

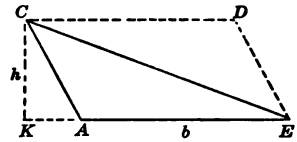
10. Construct a parallelogram equivalent to $\frac{5}{2}$ of a given parallelogram.

11. On the same base with a parallelogram construct a rhombus equal in area to the parallelogram.

12. Construct a rhombus equal in area to a given parallelogram and having each of its sides equal to a given straight line. Is it always possible?

THE TRIANGLE

362. Theorem. *The area of a triangle is equal to half the product of its base and its altitude.* $A = \frac{1}{2}bh$.



Given $\triangle ACE$ with base b and altitude h.

To prove area of $\triangle ACE = \frac{1}{2}bh$.

Proof. Construct $\square ACDE$, and draw altitude h.

$\triangle ACE = \frac{1}{2}ACDE$.	§ 154
Area of $ACDE = bh$.	§ 355
$\therefore$ area of $\triangle ACE = \frac{1}{2}bh$.	Why?

363. Theorem. *The area of a triangle equals half the area of a parallelogram having the same base and altitude as the triangle.*

364. Theorem. **The base of a triangle equals twice its area divided by its altitude.* $b = 2A \div h$.

365. Theorem. *The altitude of a triangle equals twice its area divided by its base.* $h = 2A \div b$.

366. Theorem. *Triangles having equal bases and equal altitudes are equivalent.*

367. Theorem. *Two triangles having equal bases are to each other as their altitudes.*

368. Theorem. *Two triangles having equal altitudes are to each other as their bases.*

369. Theorem. *Any two triangles are to each other as the products of their bases and their altitudes.*

---

*Here, of course, by area, base, and altitude are meant their numerical measures in each case. The numerical measure is always an *abstract number*.

EXERCISES

1. In Fig. 1, $AB \parallel CD$. Prove that $\triangle CED$, CHD, and CKD are equivalent.

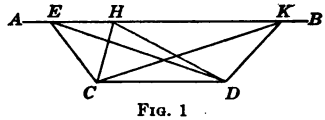


FIG. 1

2. Prove that a median of a triangle divides the triangle into two equivalent triangles.

3. In Fig. 2, $AE \parallel BD$. Prove that $\triangle ACD$ and BCE are equivalent.

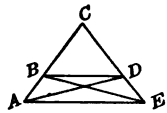


FIG. 2

4. Prove that a parallelogram is divided into four equivalent triangles by its diagonals.

5. The area of a polygon circumscribed about a circle equals half the radius of the circle multiplied by the perimeter of the polygon.

6. If FK is a median of $\triangle EFH$, prove that $\triangle EOF$ and HOF are equivalent. (Fig. 3.)

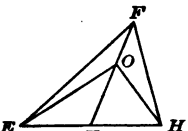


FIG. 3

7. Two equivalent triangles ABC and DBC are on opposite sides of the same base. Prove that AD is bisected by BC or BC produced.

8. In Fig. 4, $ABCD$ is a parallelogram, and P is a point in the diagonal AC. Prove that triangles PAB and PAD are equivalent.

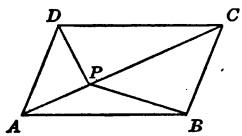


FIG. 4

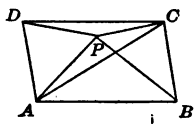


FIG. 5

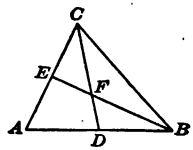


FIG. 6

9. If P is a point within $\square ABCD$, prove that $\triangle PAC$ equals the difference between $\triangle PAB$ and PAD. (Fig. 5.)

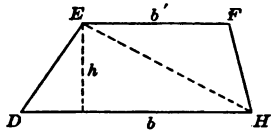
10. The medians BE and CD of $\triangle ABC$ intersect at F. Prove that triangle BFC is equivalent to quadrilateral $AEFD$. (Fig. 6.)

11. What is the locus of the vertices opposite the base of all triangles having a given fixed base and a given area?

12. What is the locus of the intersections of the diagonals of all parallelograms that have a common base and the same area?

THE TRAPEZOID

370. Theorem. ***The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.*** $A = \frac{1}{2}h(b+b')$.



Given the trapezoid $DEFH$ with altitude h and bases b and b'.

To prove area of trapezoid $DEFH = \frac{1}{2}h(b+b')$.

Proof. Draw the diagonal EH.

Area of $\triangle EFH = \frac{1}{2}b'h$.	§ 362
Area of $\triangle DEH = \frac{1}{2}bh$.	Why?
$\therefore$ area of trapezoid $DEFH = \frac{1}{2}h(b+b')$.	Why?

371. Theorem. ***The area of a trapezoid equals the product of its altitude and its median.***

Show that the median $= \frac{1}{2}(b+b')$.

EXERCISES

COMPUTATIONS

1. Find the area of a triangle having a base of 4 ft. 7 in. and an altitude of 8 ft. 6 in. Of one having a base of 5.4 ft. and an altitude of $9\frac{1}{2}$ in.

2. Find the altitude of a triangle having an area of 160 sq. ft. and a base of 20 ft. Of one having an area of 27.3 sq. in. and a base of 8.1 in.

3. In a trapezoid, A stands for the area, b and b' respectively for the bases, and h for the altitude.

(1) Given $b = 10$ in., $b' = 7$ in., $h = 8$ in.; find A.
(2) Given $b = 3\frac{3}{8}$ in., $b' = 6\frac{7}{8}$ in., $h = 5\frac{3}{4}$ in.; find A.
(3) Given $b = 7$ in., $b' = 9$ in., $A = 56$ sq. in.; find h.
(4) Given $b = 18$ ft., $h = 10$ ft., $A = 150$ sq. ft.; find b'.

4. The sides of a triangle are 7 in., 10 in., and 14 in. respectively. Is its shape fixed? Is its area fixed or may it be changed?

5. The four sides of a quadrilateral are 29 in., 31 in., 42 in., and 37 in., respectively. Can its area be found? Why?

6. The two bases of a trapezoid are 14 and 10 respectively and the other two sides are 6 and 5. Does it have a definite area?

7. Find the area of a board that is 14 ft. long and has parallel ends respectively 18 in. and 12 in.

8. Find the number of square feet in the floor of the room shown in Fig. 1.

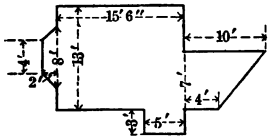


FIG. 1

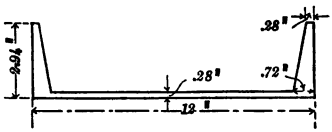
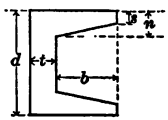


FIG. 2

9. Find the area of the cross section of the channel iron shown in Fig. 2.

Ans. 6.02 sq. in.

10. Write a formula for finding the area of the cross section of a channel iron, using the letters as given in the figure. *Ans.* $A = td + b(s + n)$.

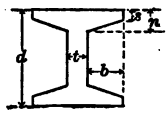


11. Using the formula derived in the preceding Exercise, find the areas of the cross sections of channel irons of the following dimensions:

No.	d	t	b	n	s	*Ans.*, A
(1)	3 in.	.17 in.	1.24 in.	.38 in.	.17 in.	1.19 sq. in.
(2)	8 in.	.22 in.	2.04 in.	.56 in.	.22 in.	3.35 sq. in.
(3)	15 in.	.40 in.	3.00 in.	.90 in.	.40 in.	

12. Write a formula for finding the area of the cross section of an I-beam, using the letters as given in the figure.

Ans. $A = dt + 2b(n + s)$.



13. Using the formula derived in the preceding Exercise, find the areas of the cross sections of I-beams of the following dimensions:

No.	d	t	b	s	n	*Ans.*, A
(1)	5 in.	.21 in.	1.395 in.	.21 in.	.44 in.	2.86 sq. in.
(2)	9 in.	.29 in.	2.02 in.	.29 in.	.627 in.	6.31 sq. in.
(3)	15 in.	.80 in.	2.80 in.	.80 in.	1.27 in.	

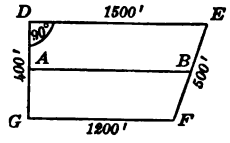
14. The base of a triangle is 3 in. longer than the altitude, and the area is 44 sq. in. Find the lengths of the base and altitude.

15. The altitude of a triangle is 3 times the base, and the area is $37\frac{1}{2}$ sq. in. Find the lengths of the base and altitude. *Ans.* 5 in. and 15 in.

16. The difference between the base and the altitude of a triangle is 6 in., and their sum is 36 in. Find the area of the triangle. *Ans.* $157\frac{1}{2}$ sq. in.

17. The altitude of a triangle is 7 in. longer than the base. If the altitude is decreased by 4 in. and the base increased by 6 in., the area is increased by 25 sq. in. Find the altitude and base.

**18.* The figure is a trapezoid with dimensions as given. Find the distance from the longer base DE to the line AB which is drawn parallel to DE and divides the area into two equal parts.



Ans. 188.93 ft.

CONSTRUCTIONS

1. Construct a triangle equivalent to a given triangle and having one of its sides equal to a given line.

2. Construct a triangle equivalent to a given triangle and having one of its angles equal to a given angle.

3. Divide a triangle into four equivalent triangles by lines through one of its vertices.

4. Construct a triangle equivalent to a given square.

5. Construct a triangle equivalent to a given triangle AHK and having its base double the length AH.

6. Construct a triangle equivalent to $\frac{4}{5}$ of a given triangle.

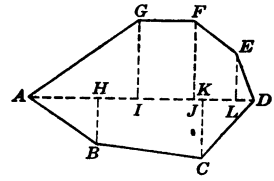
7. On the base of a given triangle, construct a right triangle equal in area to the given triangle.

8. On the base of a given triangle, construct an isosceles triangle equal in area to the given triangle.

9. On the base of a given triangle, construct a rectangle equal in area to the triangle.

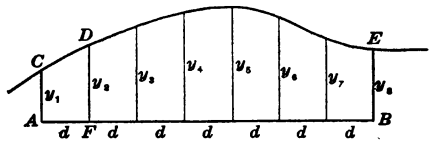
PRACTICAL METHODS

372. Practical Methods for Finding the Area of a Polygon. The area of any polygon may be found by dividing it into triangles and finding the sum of the areas of the triangles.



The area of a polygon may also be found by drawing the longest diagonal and drawing perpendiculars to this diagonal from the vertices. This will divide the polygon into triangles and trapezoids or rectangles, of which the areas can be found. The sum of these areas is the area of the polygon.

373. Trapezoidal Rule. An area between a curved line and a straight line is sometimes found approximately as follows:



(1) Divide the straight line AB, called the base line, into a number of equal parts of length d.

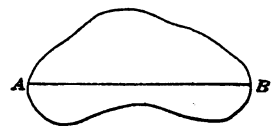
(2) Measure the lines, called **offsets,** drawn perpendicular to AB from the points of division to the curved line, their lengths being y_1, y_2, y_3, etc.

(3) Consider each part, as $ACDF$, a trapezoid and find the sum of their areas, which is the area required, approximately.

Show that this gives the following rule which is known as the *trapezoidal rule*:

To half the sum of the first and last offsets add the sum of all the other offsets, and multiply this sum by the common distance between the offsets.

By drawing a straight line, as AB in the figure, through a surface bounded by an irregular line, the area of each part can be computed approximately by the trapezoidal rule.



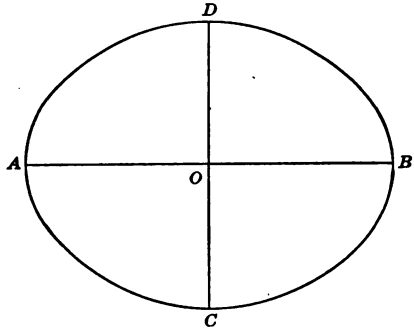
374. Areas on Square-ruled Paper. A very useful practical way to estimate the area of any plane surface is to draw it on square-ruled paper. The figure will usually be drawn to some scale that uses a side of one of the squares as a unit of length. The squares

enclosed in the figure can then be counted, and thus the area of the surface may be estimated.

A convenient way is to count as whole squares those which are entirely within or more than half within the figure, to disregard those squares which are more than half without the figure, and to count every other one of the squares that are half within and half without.

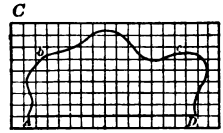
EXERCISES

1. Draw a rectangle 2 in. by 3 in. with a semicircle on one of the long sides. Measure 6 offsets from the other long side, and compute the area by the trapezoidal rule. Measure 12 offsets and compute. If the area of the figure is 9.534 sq. in., find the per cent of error in each of your results.



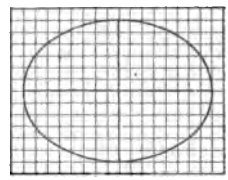
2. The figure is an ellipse that is 48 mm. long and 38 mm. wide; find its area by the trapezoidal rule. Find the per cent of error in your result if the area of the ellipse is 1432.57 sq. mm.

3. Draw an irregular polygon and compute its area.



4. Estimate the area of this figure if it is drawn on a scale of 1 in. to a side of a square.

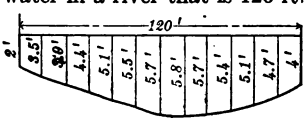
5. Take leaves of various trees and find their areas by drawing their outlines on square-ruled paper.



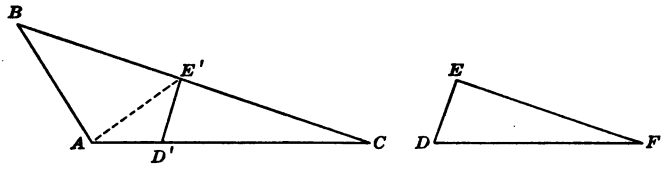
6. Estimate the area of this figure if it is drawn on a scale of 1 ft. to a side of a square. If the area of this figure is 150.8 sq. ft., what is the per cent of error in your estimate?

***7.** By the trapezoidal rule determine the area of a lake, using a map of the lake. It is necessary to know the scale to which the map is drawn. Draw the outline of the lake on square-ruled paper and determine its area.

***8.** To determine the amount of flow of water in a river that is 120 ft. wide, soundings were made for each 10 ft. of width. Thus 13 measurements were made as shown in the figure. Find the area of the cross section in square feet.



375. Theorem. *Two triangles that have an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Given $\triangle ABC$ and $\triangle DEF$ with $\angle C = \angle F$.

To prove $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AC \cdot BC}{DF \cdot EF}$.

Proof. Place $\triangle DEF$ upon $\triangle ABC$ with $\angle F$ upon $\angle C$. Then D will fall at D' and E at E', and $\triangle DEF = \triangle D'E'C$.

Draw the line AE'.

$$\frac{\triangle AE'C}{\triangle D'E'C} = \frac{AC}{D'C}. \qquad \text{§ 368}$$

$$\frac{\triangle ABC}{\triangle AE'C} = \frac{BC}{E'C}. \qquad \text{§ 368}$$

Then $$\frac{\triangle ABC}{\triangle D'E'C} = \frac{AC \cdot BC}{D'C \cdot E'C}. \qquad \text{Why?}$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{AC \cdot BC}{DF \cdot EF}. \qquad \text{Why?}$$

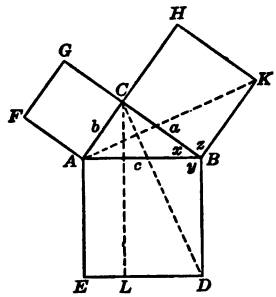
EXERCISES

1. Given two triangles with an angle of one equal to an angle of the other, and the including sides of one 10 in. and 15 in., and of the other 5 in. and 9 inches. Find the ratio of their areas.

2. If an angle of a triangle is unchanged, but each of the two including sides is doubled, how is the area changed?

THEOREM OF PYTHAGORAS

376. Theorem. ***The square constructed on the hypotenuse of a right triangle is equivalent to the sum of the squares constructed on the other two sides.***



Given rt.$\triangle ABC$, the hypotenuse c, and the other two sides, a and b, with the squares EB, CK, and FC constructed on the three sides.

To prove $\square EB = \square CK + \square FC$, or $c^2 = a^2 + b^2$.

Proof. Draw $CL \parallel BD$, and draw AK and CD.

ACH and BCG are straight lines. Why?

In $\triangle ABK$ and DBC, $AB = DB$ and $BK = BC$. Why?

$\angle x + \angle z = \angle x + \angle y$. Why?

Or $\angle ABK = \angle DBC$.

Then $\triangle ABK = \triangle DBC$. Why?

Also $\triangle ABK = \frac{1}{2}\square CK$. § 363

And $\triangle DBC = \frac{1}{2}\square LB$. Why?

$\therefore \square CK = \square LB$. Why?

In a similar manner, it may be proved that

$\square FC = \square AL$.

But $\square EB = \square LB + \square AL$. Why?

$\therefore \square EB = \square CK + \square FC$. Why?

Or $c^2 = a^2 + b^2$.

377. Theorem. *The square on either side of a right triangle is equivalent to the square on the hypotenuse minus the square on the other side.* $a^2 = c^2 - b^2$, and $b^2 = c^2 - a^2$.

378. Historical Note. The theorem of § 376 is known as the Pythagorean Proposition, and is so called because it is said to have been first proved by Pythagoras, a famous Greek mathematician who lived about 500 B. C. This proposition is the 47th of Euclid's Elements.

The proposition in a special form was known to the Egyptians as early as 2000 B. C. It was also known to the Chinese and Babylonians at a very early date.

It is not known how Pythagoras proved the theorem. The proof here given is attributed to Euclid, who lived about 300 B. C.

Because of its frequent use in mathematics and because of its numerous practical applications, the Pythagorean theorem is perhaps the most important one of plane geometry. A large number of different proofs have been devised for this famous proposition.

EXERCISES

THEOREMS

1. If a right triangle is isosceles, the area of the square on the hypotenuse is equal to four times the area of the triangle.

2. If one angle of a right triangle is 60°, the area of the square on the hypotenuse is equal to four times the area of the square on the shorter side.

3. If a perpendicular is drawn from the vertex of a triangle to its base, the difference of the squares on the segments of the base equals the difference of the squares on the other two sides.

4. The sum of the squares on the diagonals of a rhombus equals the sum of the squares on the four sides.

5. In an isosceles right triangle, prove that the sides have the ratios $1 : 1 : \sqrt{2}$.

6. In a right triangle having acute angles of 30° and 60°, prove that the sides have the ratios $2 : 1 : \sqrt{3}$.

7. Show that the following rules are true:

(1) The diagonal of a square equals a side multiplied by $\sqrt{2}$.

(2) The side of a square equals half of a diagonal multiplied by $\sqrt{2}$.

(3) The altitude of an equilateral triangle equals half of a side multiplied by $\sqrt{3}$.

NOTE. Because of their frequent use in practical applications these three rules are well worth remembering.

*8. Prove the Pythagorean theorem by means of the adjoining figures. In Figs. 4, 5, and 6, use algebraic methods. In Fig. 6, note that $ACDE$ is a trapezoid and equate areas.

NOTE. Fig. 3 is a proof by Bhaskara, a mathematician of India who lived about 1150 A.D. Fig. 6 is President Garfield's proof.

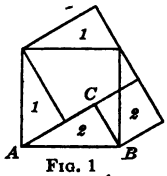


FIG. 1

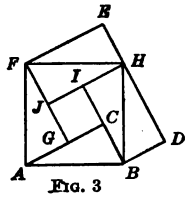


FIG. 3

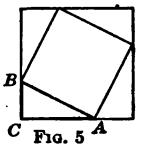


FIG. 5

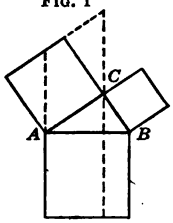


FIG. 2

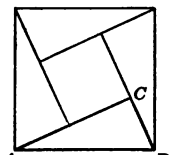


FIG. 4

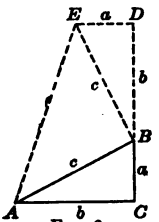


FIG. 6

COMPUTATIONS

In the following Exercises, A, a, b, and c stand for the numerical measures of the area, altitude, base, and hypotenuse, respectively, of a right triangle.

1. Given $c^2=a^2+b^2$, solve for each letter in terms of the others.

2. Show that $a^2=(c+b)(c-b)$, and $b^2=(c+a)(c-a)$.

NOTE. The relations in Exercise 2 are often convenient in computation.

3. Given $a=7$ and $b=9$; find c and A.

4. Given $c=17$ and $a=13$; find b and A.

5. Given $A=102.5$ and $a=13.4$; find b and c.

6. Given $A=30$ and $c=13$; find a and b.

7. Find the diagonal of a square whose sides are each 8 in.

8. Find the side of a square whose diagonal is 16 in.

9. Compute the altitude and area of an equilateral triangle of which a side is 3 in.

10. If the side of an equilateral triangle is s, find its area. *Ans.* $\frac{1}{4}s^2\sqrt{3}$.

11. If the area of an equilateral triangle is given by the formula $\frac{1}{4}s^2\sqrt{3}$, find the side of an equilateral triangle which has an area of 100 sq. in.

12. In Fig. 1, find the length of the brace AB, and the distance from C to B.

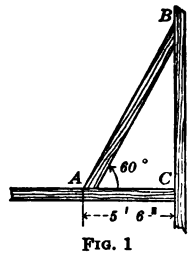


FIG. 1

13. The hypotenuse of a right triangle is 8 ft. and one side is 5 ft. Show that the equilateral triangle made on the hypotenuse is equal to the sum of the equilateral triangles made on the other two sides.

14. The area of a rectangular lawn is 5525 sq. m., and the length of one of its sides is 85 m. Find the length of its diagonal in meters to three decimal places.

Ans. 107.005 m.

***15.** Find the diagonal of a cube 9 ft. on an edge. (Fig. 2.) *Ans.* $9\sqrt{3}$ ft.

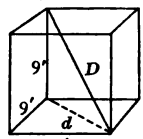


FIG. 2

***16.** In the gambrel roof shown in section in Fig. 3, find the lengths of rafters and parts whose dimensions are not given.

Ans. $AB = 15$ ft. $7\frac{3}{4}$ in.; $DB = 15$ ft. $2\frac{3}{4}$ in.; $BC = 14$ ft. $3\frac{3}{4}$ in. (All to the nearest $\frac{1}{4}$ in.)

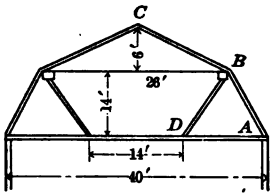


FIG. 3

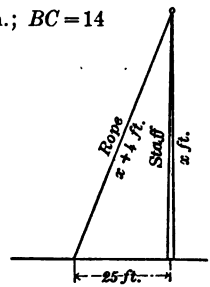


FIG. 4

17. The height of the flagstaff shown in Fig. 4 is unknown; but it is noticed that the flag rope which is 4 ft. longer than the staff, when stretched out just reaches the ground at a point 25 ft. from the foot of the staff. If the ground is level, find the height of the staff.

Ans. $76\frac{1}{8}$ ft.

18. A tree, CD in Fig. 5, 75 ft. high, breaks at a point B, and the end D strikes the ground at A, a distance of 40 ft. from C. Find the length BD that was broken off. *Ans.* $48\frac{1}{8}$ ft.

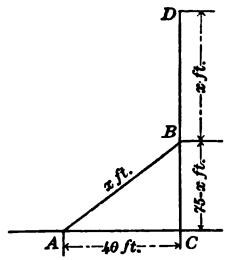


FIG. 5

19. The differences between the hypotenuse and the two sides of a right triangle are 3 and 6 feet respectively. Find the sides and the area of the triangle. *Ans.* 15 ft., 12 ft., 9 ft., and 54 sq. ft.

PROJECTION

379. If from a point P, Fig. 1, a perpendicular PQ is drawn to any straight line RS, the foot of the perpendicular, Q, is said to be the **projection** of P upon RS.

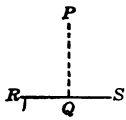


FIG. 1

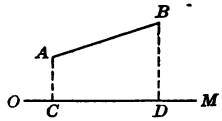


FIG. 2

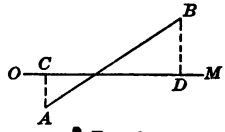
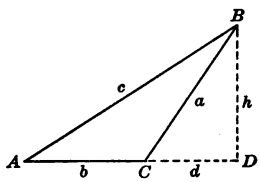


FIG. 3

The **projection of a line** upon a given straight line is the portion of the given line lying between the projections of the ends of the line.

In Figs. 2 and 3, CD is the projection of AB upon OM.

380. Theorem. *In any obtuse triangle, the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the other two sides, plus twice the product of one of those sides and the projection of the other side upon it.*



Given the obtuse triangle ABC, the angle ACB being obtuse, and d and AD being the projections of a and c, respectively, upon AC.

To prove $c^2=a^2+b^2+2bd$

Proof. $AD=b+d$

Then $\overline{AD}^2=b^2+2bd+d^2$. Why?

Adding h^2 to both members of this equation,

$\overline{AD}^2+h^2=b^2+2bd+d^2+h^2$.

But $\overline{AD}^2+h^2=c^2$, and $d^2+h^2=a^2$. § 376

$\therefore c^2=a^2+b^2+2bd$. Why?

Similarly, if b' is the projection of b upon CB, it can be proved that

$$c^2=a^2+b^2+2ab'.$$

381. Theorem. *In any triangle, the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

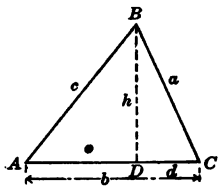


FIG. 1

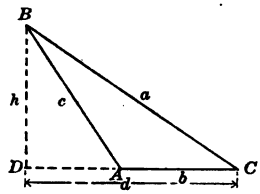


FIG. 2

Given the triangle ABC, having an acute angle C, and d the projection of a upon AC.

To prove $c^2=a^2+b^2-2bd$.

Proof. $AD=b-d$, Fig. 1, or $AD=d-b$, Fig. 2.

In both cases, $\overline{AD}^2=b^2-2bd+d^2$. Why?

Adding h^2 to both members of this equation,

$$\overline{AD}^2+h^2=b^2-2bd+d^2+h^2.$$

But $\overline{AD}^2+h^2=c^2$, and $d^2+h^2=a^2$. § 376

$$\therefore\ c^2=a^2+b^2-2bd.$$ Why?

Similarly, if b' is the projection of b upon CB, it can be proved that

$$c^2=a^2+b^2-2ab'.$$

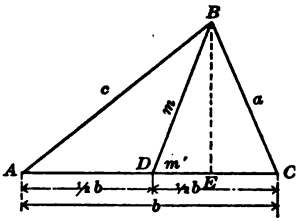
NOTE. It should be noticed that the theorems of § 380 and § 381 become the same as the theorem of § 376, if the angle opposite side c changes into a right angle. Show that this is true.

382. Theorem. *An angle of a triangle is acute, right, or obtuse according as the square of the opposite side is less than, equal to, or greater than the sum of the squares of the other two sides.*

383. Theorem. *If any median of a triangle be drawn to a side:*

(1) *The sum of the squares of the other two sides is equal to twice the square of the median plus twice the square of half the bisected side.*

(2) *The difference of the squares of the other two sides is equal to twice the product of the bisected side and the projection of the median upon that side.*



Given $\triangle ABC$, with the median from B.

Let a, b, c, and m represent the three sides and the median respectively. Let m' represent the projection of the median upon AC. Also let c be greater than a.

To prove (1) $a^2+c^2=2m^2+2(\frac{1}{2}b)^2$;

(2) $c^2-a^2=2bm'$.

Proof. Draw the altitude BE.

Since $c>a$, $\angle ADB$ is obtuse, $\angle CDB$ is acute, and D is between A and E. Why?

$$c^2=m^2+(\tfrac{1}{2}b)^2+2(\tfrac{1}{2}b)m', \qquad \S\ 380$$

and $$a^2=m^2+(\tfrac{1}{2}b)^2-2(\tfrac{1}{2}b)m'. \qquad \S\ 381$$

Adding, $a^2+c^2=2m^2+2(\frac{1}{2}b)^2$.

Subtracting, $c^2-a^2=2bm'$.

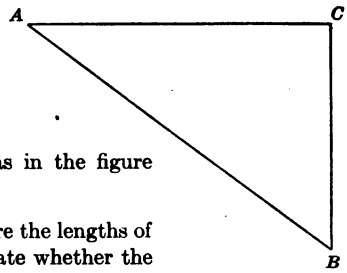
NOTE. This theorem enables one to compute the medians of a triangle when the three sides are given.

EXERCISES

1. Prove that in any triangle one of whose angles is 60°, the square of the side opposite the 60° angle is equivalent to the sum of the squares of the other two sides minus the product of these two sides.

2. Prove that in any triangle one of whose angles is 120°, the square of the side opposite the 120° angle is equivalent to the sum of the squares of the other two sides plus the product of these sides.

3. In triangle ABC, $AC=4$ cm., $BC=3$ cm., and $AB=5$ cm. Show that $\angle C$ is a right angle.



4. Show how a right angle could be laid out with a rope 12 ft. long by passing it around three stakes set as in the figure of Exercise 3.

5. The following sets of numbers are the lengths of the sides of triangles. In each case state whether the largest angle is right, acute, or obtuse. (*a*) 2, 4, 5; (*b*) 9, 12, 15; (*c*) 8, 11, 14; (*d*) 4, 5, 6; (*e*) 15, 20, 25; (*f*) 13, 17, 18.

6. In a triangle, the sides are $a=7$, $b=9$, $c=10$. Find the projection of a upon c.

SOLUTION. Let d be the projection.

$$b^2=a^2+c^2-2cd. \qquad \S\ 381$$

Then $$2cd=a^2+c^2-b^2.$$

$$\therefore d=\frac{a^2+c^2-b^2}{2c}$$

7. In the triangle of Exercise 6, find the altitude drawn to the side c.

SUGGESTION. Let h be the altitude; then $h^2=a^2-d^2$. § 377

8. In the triangle whose sides are $a=6$, $b=8$, and $c=12$, find the projection of b upon c, and also the altitude drawn to the side c.

***9.** Prove that a straight line is never less than its projection on another straight line. When are they equal? When is the projection of a straight line upon another straight line a point only?

***10.** Discuss the theorem of § 383 if $c=a$.

***11.** Write formulas analogous to (1) of theorem § 383 for the medians upon a and c.

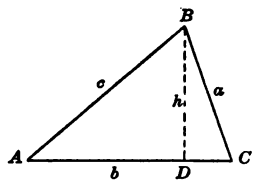
***12.** Solve the formula of theorem § 383 and obtain the following:

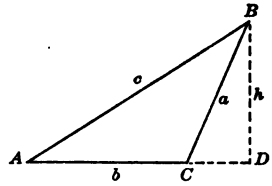
$$m=\tfrac{1}{2}\sqrt{2(a^2+c^2)-b^2}.$$

Write analogous formulas for the medians on the other two sides.

***13.** Use the formulas of the previous Exercise and find the three medians of a triangle having sides of 13 in., 16 in., and 12 in. Give answers correct to two decimal places. *Ans.* 12.56 in.; 9.62 in.; 13.29 in.

14. Derive a formula for the altitude of a triangle in terms of its sides.





At least one of the angles A or C is acute. Let A be an acute angle. The altitude designated by h will fall within the triangle if C is acute, and without if C is obtuse. In either case

$$a^2 = b^2 + c^2 - 2b \cdot AD. \qquad \text{§ 381}$$

Then $$AD = \frac{b^2+c^2-a^2}{2b}.$$

In $\triangle ABD$, $$h^2 = c^2 - \overline{AD}^2. \qquad \text{Why?}$$

$$= c^2 - \left(\frac{b^2+c^2-a^2}{2b}\right)^2$$

$$= \frac{4b^2c^2 - (b^2+c^2-a^2)^2}{4b^2}$$

$$= \frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4b^2}$$

$$= \frac{[(b^2+2bc+c^2)-a^2]\,[a^2-(b^2-2bc+c^2)]}{4b^2}$$

$$= \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}.$$

Let $$s = \tfrac{1}{2}(a+b+c).$$

Then $$2s = a+b+c,$$

$$2(s-a) = a+b+c-2a = b+c-a,$$

$$2(s-b) = a+b+c-2b = a-b+c,$$

$$2(s-c) = a+b+c-2c = a+b-c.$$

Substituting these values,

$$h^2 = \frac{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)}{4b^2}.$$

$$\therefore h = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

15. Derive formulas for the altitudes upon the other sides.

16. Derive a formula for computing the area of a triangle when the three sides only are given.

SUGGESTION. By § 362, $A=\frac{1}{2}bh$. Substitute value for h and obtain,

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

HISTORICAL NOTE. This formula for the area of a triangle was discovered by Hero of Alexandria, who probably lived in the first century A.D.

17. Compute the altitudes of a triangle, if $a=25$, $b=36$, and $c=29$.

Ans. Altitude upon $a=28.8$.

18. Compute the areas of the following triangles:

(1) $a=40$, $b=30$, $c=20$. *Ans.* 290.47 sq. in.

(2) $a=27$, $b=61$, $c=34$.

(3) $a=49$, $b=23$, $c=15$.

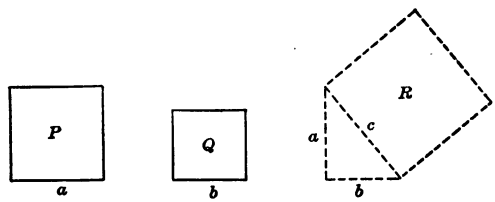
*19. Two sides of a parallelogram are 15 in. and 22 in. respectively, and one diagonal is 18 in. Compute the other diagonal and the area.

Ans. 33.08 in.; 268 sq. in.

TRANSFORMATIONS AND CONSTRUCTIONS

384. Definition. To **transform** a figure is to construct another that is equivalent to it.

385. Problem. *To construct a square that is equivalent to the sum of two given squares.*

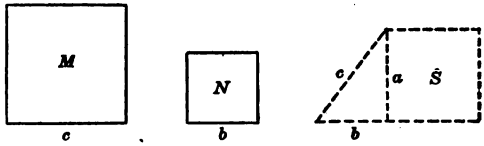


Given squares P and Q.

To construct a square equivalent to the sum of P and Q.

Construction. Construct a rt.△ having a and b as sides. Upon the hypotenuse c of this triangle construct the square R. Why is this the required square?

386. Problem. *To construct a square that is equivalent to the difference of two given squares.*



Given squares M and N, with sides c and b, respectively.

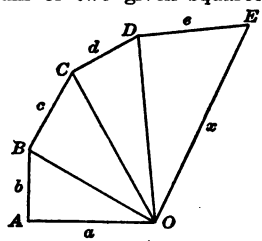
To construct a square equivalent to M minus N.

Construction. Construct a rt.$\triangle$ having base b and hypotenuse c. Upon the altitude a of this triangle construct the square S. Why is this the required square?

EXERCISES

1. Construct a square equivalent to the sum of two given squares with sides 3 cm. and 5 cm. respectively.

2. Construct a square with side x equivalent to the sum of the squares of sides a, b, c, d, and e.



SUGGESTION. The construction is shown in the figure, where angles at A, B, C, and D are right angles. Why does this construction give the required square?

3. Four air pipes that are square in cross section and 6, 8, 10, and 12 inches respectively on an edge, join and continue as one pipe, which, also, is square in cross section. Determine, by constructing to scale, the edge of the cross section of the continuation pipe having the same area as the four pipes. Test the work by computing.

4. Compare with a given square the squares constructed on its diagonal and on half its diagonal.

5. Construct a line $\sqrt{2}$ in. long. Construct a square containing 2 sq. in.

6. Construct a line $\sqrt{3}$ in. long. Construct a square containing 3 sq. in.

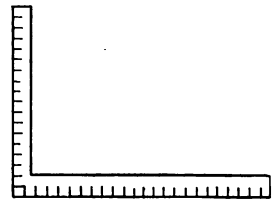
7. Construct a line $\sqrt{5}$ in. long. Construct a square containing 5 sq. in.

8. Draw two squares having sides of $1\frac{1}{4}$ in. and $2\frac{1}{4}$ in., respectively. Make a third square equivalent to the difference of the first two. Check by computation.

9. Construct a square having each side equal to 2 in. Construct a second square that is double the first. Measure and check by computation.

10. Construct a square having one side 3.2 cm. Construct a second square three times the area of the first. Measure the sides and check by computation.

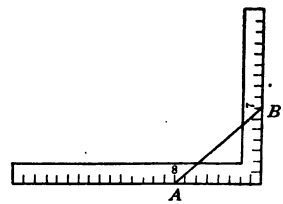
387. The Steel Square. One of the most useful mechanical instruments is the **steel square,** or **carpenter's square,** shown in the figure. It is made in various sizes, but the most common size is with the longer arm, called the **body, blade,** or **stock,** 24 inches in length and 2 inches wide; and with the shorter arm, called the **tongue,** 16 or 18 inches in length and $1\frac{1}{2}$ inches wide. The blade and the tongue form a right angle.



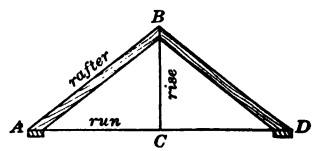
Many books have been written on the uses of the steel square. The principles involved in using the steel square are mainly those involved in the solution of the right triangle, and in similar triangles (§ 420). One who understands the right triangle and similar triangles can devise many uses for the steel square, and can readily see the principles underlying the various uses of this instrument given in treatises on the steel square.

EXERCISES

1. To find the length of a brace forming the hypotenuse of a right triangle that has a base of 8 ft. and an altitude of 7 ft., one could take 8 in. on the blade and 7 in. on the tongue of a square, as shown in the figure, and then measure the distance AB. This will give the length of the brace to scale. Explain and state theorem involved.

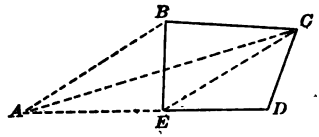


2. Find by means of the carpenter's square the length of a rafter that has a run of 15 ft. and a rise of 10 ft.



3. Show how any number of squares may be added by means of the steel square. That is, find the side of a square that is equivalent to the sum of any number of squares.

388. Problem. *To transform a given quadrilateral into an equivalent triangle.*



Given the quadrilateral $BCDE$.

To transform $BCDE$ into an equivalent triangle.

Construction. Draw the diagonal EC. Draw $BA \parallel EC$ meeting DE produced at A, and draw AC. Then $\triangle ACD$ is the required triangle.

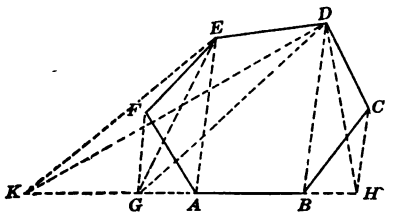
Proof.

$\triangle BCE = \triangle ACE$. § 366

$\triangle BCE + \triangle CDE = \triangle ACE + \triangle CDE$. Why?

$\therefore \triangle ACD = BCDE$.

389. Problem. *To transform a given polygon into an equivalent triangle.*



Given the polygon $ABCDEF$.

To transform the polygon into an equivalent triangle.

Construction. The diagonal BD is drawn and $\triangle BDH$ is constructed equivalent to $\triangle BDC$ as in § 388.

Similarly $\triangle AGE$ is constructed equivalent to $\triangle AFE$.

Then quadrilateral $GEDH$ = polygon $ABCDEF$.

Construct $\triangle KDH$ = quadrilateral $GEDH$. § 388.

$\therefore \triangle KDH$ = polygon $ABCDEF$.

EXERCISES

1. Transform a square 3 cm. on a side into an equivalent triangle.

*2. Draw a pentagon and transform it into an equivalent triangle.

390. Problem. *To construct a square equivalent to a given rectangle or parallelogram.*

Given the rectangle $ABCD$.

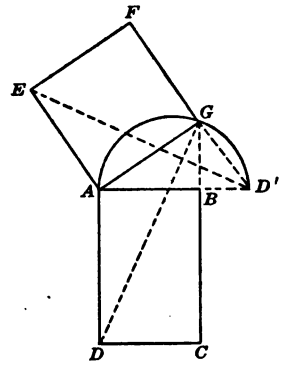
To construct an equivalent square.

Construction. Produce a shorter side as AB of $\square ABCD$ to D', making $AD' = AD$, and on AD' as a diameter describe a semicircle.

Produce CB to meet the semicircle at G.

Draw AG and construct the square $AEFG$.

Then $AEFG$ is the required square.



Proof. Draw ED' and DG.

$\triangle AED' = \triangle AGD$. Why?

But $\square AEFG = 2\triangle AED'$, Why?

and $\square ABCD = 2\triangle AGD$. Why?

$\therefore \square AEFG = \square ABCD$.

The student should discuss the problem for the parallelogram.

391. Problem. *To construct a square equivalent to a given triangle.*

392. Problem. *To construct a square equivalent to a given polygon.*

SUGGESTION. First construct a triangle equivalent to the given polygon and then construct a square equivalent to the triangle.

EXERCISES

CONSTRUCTIONS

1. Construct a square equivalent to a given quadrilateral.
2. Construct a square equivalent to the sum of two triangles.
3. Construct a square containing 15 sq. cm.

SUGGESTION. The square is equivalent to a rectangle 5 cm. by 3 cm

4. Construct a square equivalent to $\frac{3}{7}$ of a square 7 cm. on an edge.

5. Construct a square equivalent to $\frac{5}{8}$ of a given square.

6. Construct a square equivalent to the difference in the areas of two given pentagons.

7. Construct a right triangle that is equivalent to a given square.

8. Construct a right triangle that is equivalent to the difference of any two triangles.

9. Construct a square that is equivalent to $\frac{3}{5}$ of a given pentagon.

10. Construct a triangle whose base, altitude, and area are equal to those of a given triangle.

11. Construct the circle with center O in Fig. 1, that is tangent to the semicircle and to the equilateral arch.

SUGGESTION. Find the radius by using rt.$\triangle KOE$.

Let x be the radius.

Then $a^2+(a+x)^2=(2a-x)^2$. Why?

Find the value of x in this equation.

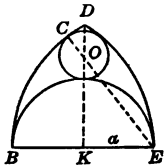


FIG. 1

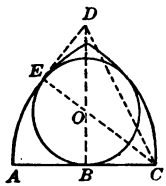


FIG. 2

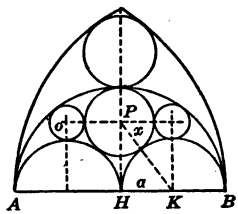


FIG. 3

12. To inscribe a circle in an equilateral arch formed on AC. (Fig. 2.)

SUGGESTION. Construct $BD=AC$, and DE tangent to the arc.

$\triangle EDC=\triangle BDC$. Why?

$\triangle OED=\triangle OBC$. Why?

13. The circle whose center is P is tangent to the three semicircles as shown in Fig. 3. Find its radius, and make the construction as in the figure. Also construct the circle with center O.

SUGGESTION. $HP=2a-x$. $\therefore (2a-x)^2+a^2=(a+x)^2$.

NOTE. The method of construction in Exercises 11 and 13 shows the use of algebra in analyzing a construction.

14. A hot air pipe that is 10 in. square in cross section changes into a pipe of the same capacity and having its cross section a rectangle 4 in. wide. Find the length of the cross section of the second pipe.

GENERAL EXERCISES

THEOREMS

1. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other two sides.

2. Prove that two quadrilaterals are equivalent when they have the following parts of one respectively equal to the corresponding parts of the other:

(1) Four sides and one diagonal.
(2) Four sides and one angle.
(3) Two adjacent sides and three angles.
(4) Three sides and two included angles.

3. AB and ECD are two parallel straight lines; BF and DF are drawn parallel to AD and AE respectively. Prove that $\triangle ABC$ and DEF are equal in area.

SUGGESTION. Draw AD, AF, and BD.

4. Prove that the difference of the squares on two straight lines equals the rectangle of which the length is the sum of the straight lines, and the breadth is the difference of the straight lines.

5. Show that the sides of a right triangle may be represented by

$$n, \frac{n^2-1}{2}, \text{ and } \frac{n^2+1}{2},$$

if n has such a value that the sides are positive.

6. Show that the sides of a right triangle may be represented by $2n$, n^2-1, and n^2+1. Also by p^2+q^2, p^2-q^2, and $2pq$.

7. State in integers the sides of four different right triangles.

8. If two sides of a triangle are unequal, the median drawn to the shorter side is greater than the median drawn to the longer side.

9. Prove that the following method of bisecting any quadrilateral by a line drawn from a vertex is correct: Let $ABCD$ be any quadrilateral. Draw AC and BD. Bisect BD at E. Draw EF parallel to AC and meeting BC in F. Draw AF. Then AF bisects the quadrilateral.

SUGGESTION. Draw AE and EC.

10. Inscribe a circle in a triangle ABC, touching its sides at the points D, E, and F. Draw three circles with centers at A, B, and C respectively, each tangent to the other two externally. Show that the points of tangency are D, E, and F.

11. In any parallelogram, the squares on the diagonals are together equal to the squares on the sides, or to twice the squares on adjacent sides.

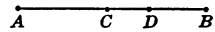
(This theorem is probably due to Gregory of St. Vincent [1584-1657].)

***12.** In any quadrilateral, the sum of the squares on the sides is equal to the sum of the squares on the diagonals plus four times the square on the line joining the middle points of the diagonals.

(Leonard Euler [1707-83] was the first to discover this theorem.)

13. If there are two straight lines AB and CD, and if CD is divided at E into two parts, the rectangle formed by AB and CD equals that formed by AB and CE plus the rectangle formed by AB and ED.

14. Given C, the middle point of the straight line AB, and D any other point in the line CB. Prove that:



(1) The rectangle formed by AD and DB is equal to $\overline{AC}^2-\overline{CD}^2$.

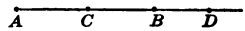
(2) $\overline{AD}^2+\overline{DB}^2=2\overline{AC}^2+2\overline{CD}^2$.

SUGGESTION. Let $AC=CB=s$, and $CD=r$.

15. Given C, the middle point of a straight line AB, and D any point in AB produced. Prove that:

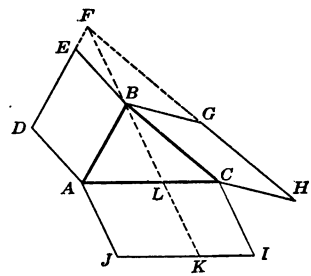
(1) The rectangle formed by AD and DB is equal to $\overline{CD}^2-\overline{AC}^2$.

(2) $\overline{AD}^2+\overline{DB}^2=2\overline{AC}^2+2\overline{CD}^2$.



Draw a diagram to illustrate these results.

***16.** ABC is any triangle. $ADEB$ and $CBGH$ are any parallelograms drawn on two sides of the triangle. DE and HG are produced to meet at F. The parallelogram $ACIJ$ is drawn with the side AJ equal and parallel to FB.

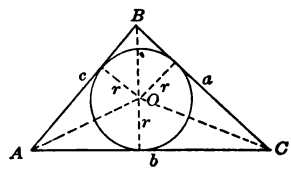


Then $\square ADEB+\square CBGH=\square ACIJ$.

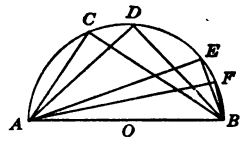
(This theorem is known as Pappus' extension of the Pythagorean theorem. Pappus lived about 300 A.D.)

SUGGESTION. Prove $\square ADEB=\square ALKJ$.

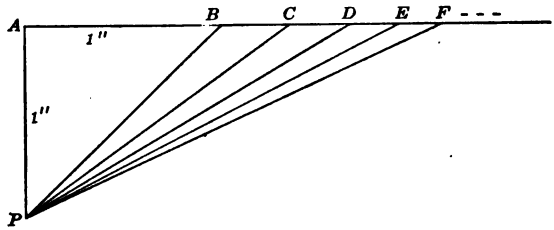
17. Prove that the area of any triangle equals half the product of its perimeter and the radius of the inscribed circle.



18. A rubber band is stretched around two pegs driven at A and B. A pencil is placed against the rubber band so as to stretch it into the form of a triangle with the pencil point on a circle of which AB is the diameter. Are the triangles formed of equal area? Of equal perimeter? What facts can you state about the variable triangle?



CONSTRUCTIONS



1. Make a construction as shown in the figure. $AP = 1$ in. $\angle A$ is a right angle. $AB = 1$ in. $AC = PB$, $AD = PC$, $AE = PD$, etc. Prove that $AC = \sqrt{2}$ in., $AD = \sqrt{3}$ in., $AE = \sqrt{4}$ in., $AF = \sqrt{5}$ in., etc.

2. Construct a rectangle of given base equal to the difference between two given squares.

3. On the same base with a given parallelogram, construct a parallelogram equal in area to the given parallelogram, and having one of its sides equal to a given straight line.

4. Trisect the area of a triangle by straight lines drawn through a vertex.

5. Divide the area of a triangle into any number of equal parts by straight lines drawn through a vertex.

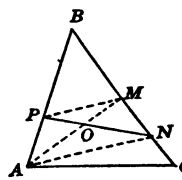
6. Bisect the area of a triangle by a straight line drawn through a given point in one of the sides.

SUGGESTION. ABC is the triangle and P the point. Draw AM a median. Draw $AN \parallel PM$.

Then PN is the required line.

$\triangle PMA = \triangle PMN$. Why?

$\triangle POA = \triangle MON$. Why?

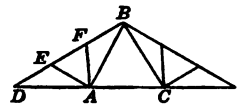


7. Trisect a given parallelogram by two straight lines drawn through one of its vertices.

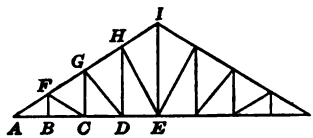
8. Draw a diagram to illustrate the equation $(a+b)^2-(a-b)^2=4ab$.

COMPUTATIONS

1. The triangular roof truss (fan truss) shown in the figure is 60 ft. long, and ABC is an equilateral triangle. The bottom member and the rafters are divided into equal segments. Find the length of each member and the angles they make with the bottom member.



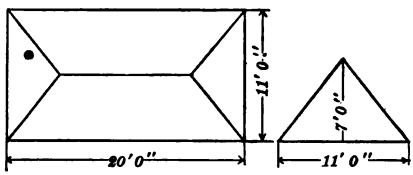
2. The triangular roof truss shown in the figure is 80 ft. long and 25 ft. high. The bottom member is divided into eight equal segments by vertical members. Find the length of the various parts.



***3.** A horizontal steam pipe with a diameter of 18 in., runs at right angles to the side of a building. It is to pass through a "third-pitch" roof. Determine the length and width of the hole the carpenter must cut in the roof. (In a "third-pitch" roof the rise is two-thirds of the run.)

***4.** The dimensional sketch in the figure shows the plan and section of a roof. What will be the area of boards required to cover the roof?

Ans. 356 sq. ft.



5. Find the area of an isosceles trapezoid whose upper base is 24 m., altitude 1200 cm., and one of whose equal sides is 1300 cm.

6. The diagonals of a rhombus are in the ratio of 3 : 5. Find these diagonals if the area of the rhombus is equivalent to the area of an equilateral triangle one of whose sides is 24 in. *Ans.* 17.3 in., 28.8 in.

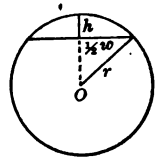
7. The vertical angle of an isosceles triangle is 120°, and the base and one of the equal sides are in the ratio of $\sqrt{3} : 1$. Find the altitude if the perimeter is 120 in. Give the result correct to two decimal places.

Ans. 16.07 in.

***8.** Given a triangle ABC having $a = 26$ mm., $b = 32$ mm., and $c = 48$ mm. Compute the lengths of the three medians of the triangle.

Ans. Median drawn to $a = 38.66$ mm.

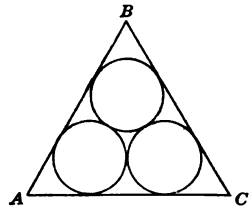
9. Given two concentric circles whose radii are R and r. Find the length of a chord of the greater that is tangent to the smaller.



*10. If r = the radius of a circle, w = the length of a chord, and h = the height of the segment cut off by the chord, derive the following formulas:

$$r=\frac{(\frac{1}{2}w)^2+h^2}{2h}, \quad w=2\sqrt{2hr-h^2}, \quad h=r-\sqrt{r^2-(\tfrac{1}{2}w)^2}.$$

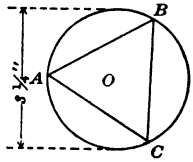
NOTE. The figure formed by an arc and its chord is called a **segment**.



*11. Three circles each tangent to the other two are enclosed in an equilateral triangle as shown in the figure. If the radius of each circle is 10 in., find the sides of the triangle. *Ans.* 54.64 in.

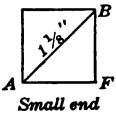
*12. Find the height of the segment cut off by a chord 6 ft. 6 in. long, in a circle of radius 5 ft. *Ans.* 1.200 ft.

*13. A segment of a circle cut off by a chord 4 ft. 6 in. long has a height of 1 ft. 10 in. Find the radius of the circle. *Ans.* 27.57− in.

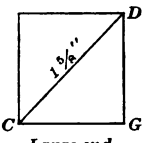


14. A round shaft has a diameter of $3\frac{1}{4}$ in. Find the length of the side of a triangular end that can be made on the shaft. Find the length of the side of a square end. *Ans.* 2.81+ in.; 2.30− in.

*15. A square taper reamer is to be made which must ream $1\frac{1}{8}$ in. at the small end and $1\frac{5}{8}$ in. at the large end. What must be the width of the flat face at each end? *Ans.* .795+ in.; 1.149+ in.



Small end



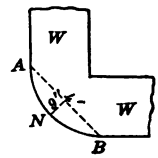
Large end

*16. If one side of a regular octagon is a, show that its area is $2a^2(\sqrt{2}+1)$. Compute the area of a regular octagon if one side is 8 in.

*17. Using the formula of the preceding Exercise, find the radius of the circle inscribed in a regular octagon one of whose sides is 6 in.

*18. Near the base of a large chimney, the outer form is a regular octagon 6 ft. on an edge. The flue is circular and 7 ft. in diameter. How thick is the thinnest part of the wall at the base? *Ans.* $3\frac{3}{4}$ ft.

19. What is the area of a pentagon circumscribed about a circle with a diameter of 30 ft. if the perimeter of the pentagon is 180 ft.?



*20. In the figure, W W is a wall with a round corner, of dimensions as given from A to B, on which a gutter is to be placed. Find the radius of the circle of which arc ANB is a part. *Ans.* 3 ft. $\frac{1}{2}$ in.

*21. In the octagonal design of a steel ceiling shown on page 101, the side of the square is 18 in. The outer octagon has twice the area of the inner one. Find their areas.

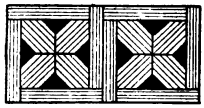


FIG. 1

22. Fig. 1 is an oak and mahogany parquetry floor, the details of which are shown in Fig. 2.

In Fig. 2, *ABCDEFGH* is a regular octagon. If $PQ = 12$ in., find the number of square inches of each kind of wood in one square if $RS = \frac{2}{3}PQ$.

Find the area of triangle *DNE*. Prove $\triangle DNE$ is an isosceles right triangle. Find area of rectangle *CN*. Find the area of the octagon *ABCDEFGH*.

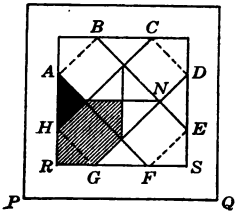


FIG. 2

ALGEBRAIC EXERCISES

1. In a right triangle the two sides about the right angle are 6 and 8 in., respectively. Calculate the altitude upon the hypotenuse.

Ans. $4\frac{4}{5}$ in.

2. The base of a right triangle is 10 ft. and the altitude upon the hypotenuse is 6 ft. Compute the area of the triangle. *Ans.* 37.5 sq. ft.

3. The altitude of an isosceles triangle is 8 ft. and its perimeter is 32 ft. Compute the sides. *Ans.* 12 ft. and 10 ft.

4. In the center of a square pool 22 ft. on a side, there stood a bamboo which extended 5 ft. above the water. When pulled over, the top of the bamboo would just reach to the center of one side of the pool. Find the depth of the water in the pool. (From a Chinese book written 2000 B.C.)

5. The sum of one side and the altitude of an equilateral triangle is $45\sqrt{3}$ in. Compute the area. *Ans.* 755.4 sq. in.

6. Compute the area of a right triangle having the sum of the sides about the right angle equal to 20 in., and the altitude upon the hypotenuse equal to 6 in.

7. Find the equal sides of an isosceles trapezoid if the bases are b_1 and b_2 and the altitude is h.

8. A rectangle is 12 in. by 5 in. Divide one of the longer sides into two parts such that the difference of the squares formed on them shall be equivalent to the area of the rectangle. *Ans.* 8.5 in. and 3.5 in.

9. Given a rectangle of sides a and b, $b>a$. Divide the side b into two parts such that the difference of the squares on the parts shall be equivalent to the area of the rectangle.

Ans. $\frac{b+a}{2}, \frac{b-a}{2}$.

Explain how this problem could be solved geometrically, that is, constructed by means of the compasses and straightedge.

10. Given three points A, B, and C in a straight line. Determine on the same straight line a point P such that, $\overline{AP}^2 = BP \cdot CP$.

11. Divide a line 20 in. long into three parts such that the first is to the second as 2 : 3, and the second is to the third as 2 : 5.

Ans. $3\frac{1}{3}$ in.; $4\frac{4}{5}$ in.; 12 in.

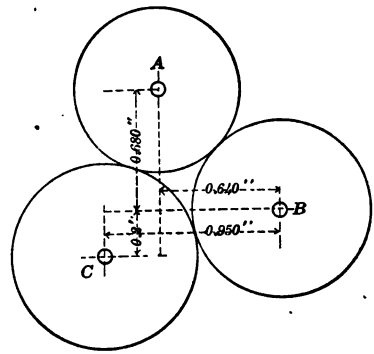
12. Divide a line 16 in. long into two parts such that the square on one of the parts shall be equivalent to the product of the whole line and the other part.

Ans. 9.89− in.; 6.11+ in.

13. Find the dimensions of a rectangle that is equivalent to a square 10 in. on a side, and has a perimeter double that of the square.

Ans. 37.32 in. by 2.68 in.

14. Three points A, B, and C are located as shown in the figure. Three discs are centered at these points, each tangent to the other two. Find the diameters of the disks.



Ans. At A, 0.8960 in.; at B, 0.9716 in.; at C, 0.9700 in.

QUESTIONS

1. Which of the following figures have areas of fixed values if the perimeters are given: square, rectangle, right triangle, right isosceles triangle, equilateral triangle, rhombus, trapezoid, parallelogram with one angle 60°, isosceles trapezoid with one angle 60°, rhombus with angle 60°?

2. Of how many of the preceding can you determine the areas?

3. What measurements are necessary in order that one may determine the area of a square? A rectangle? A rhombus? A parallelogram? A triangle? An equilateral triangle? A trapezoid? A quadrilateral? A hexagon that is not regular?

CHAPTER IV

PROPORTION AND SIMILARITY

393. Terms of a Proportion. In the proportion $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d$, the four numbers a, b, c, and d are called the **terms** of the proportion. The terms a and d are called the **extremes,** and the terms b and c are called the **means.**

394. Fourth Proportional. The fourth term of a proportion is called the **fourth proportional** to the first three terms taken in order.

In the proportion $\frac{a}{b}=\frac{c}{d}$, d is the fourth proportional to a, b, and c.

395. Mean Proportional. If the two means of a proportion are equal, this common mean is called the **mean proportional** between the two extremes.

In the proportion $\frac{a}{b}=\frac{b}{c}$, b is the mean proportional between a and c.

396. Continued Proportion. The numbers $a, b, c, d, e, \cdots$ are said to be in **continued proportion** if $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\cdots$.

397. The terms of one ratio are said to be **inversely proportional** to the terms of another ratio when the first ratio equals the reciprocal* of the second.

Thus, a and b are inversely proportional to c and d if $\frac{a}{b}=\frac{d}{c}$. For example, 3 and 5 are inversely proportional to 15 and 9, for $\frac{3}{5}=\frac{9}{15}$.

*The reciprocal of a number is 1 divided by that number. Then, the reciprocal of a fraction is the fraction inverted. Thus the reciprocal of 3 is $\frac{1}{3}$; the reciprocal of $\frac{5}{6}$ is $1\div\frac{5}{6}=\frac{6}{5}$.

FUNDAMENTAL THEOREMS

398. Theorem. *If four numbers form a proportion, the product of the extremes equals the product of the means.*

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $ad=bc$.

Proof. Multiplying both ratios by bd, $ad=bc$.

NOTE. A proportion is an equation and may be treated as such.

399. Theorem. *The mean proportional between two numbers equals the square root of their product.*

That is, if $\frac{a}{b}=\frac{b}{c}$, then $b=\sqrt{ac}$. Why?

400. Theorem. *If the two antecedents of a proportion are equal, the two consequents are equal.*

401. Theorem. *If three terms of one proportion are equal respectively to three corresponding terms of another proportion, the fourth terms are equal.*

That is, if $\frac{a}{b}=\frac{c}{x}$, and $\frac{a}{b}=\frac{c}{z}$, then $x=z$. Why?

EXERCISES

1. Find the value of x if $10:16=7:x$.
2. Find the value of x if $24:x=64:4$.
3. Find the value of x if $a:b=x:d$.
4. Find the value of x if $3:x=x:12$.
5. Find the value of x if $\frac{2x-5}{3x+4}=\frac{4x+2}{15x+36}$.
6. If x is a mean proportional between a and b, show that $a:b=a^2:x^2$.
7. Find the fourth proportional to 1, 2, and 3.
8. Find the mean proportional between each of the following pairs of numbers: (1) 5 and 20, (2) 4 and 100, (3) $2xy$ and $8xy$, (4) $a+b$ and $a-b$.

402. Theorem. ***If the product of two numbers equals the product of two other numbers, either two may be made the means and the other two the extremes of a proportion.***

Given $ad=bc.$

To prove $\frac{a}{b}=\frac{c}{d}.$

Proof. $ad=bc.$

Dividing each member by bd, $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d.$

The student should note the number of proportions that may be formed and should obtain several of them.

EXERCISES

1. Form all proportions possible from $5\times6=3\times10.$
2. Form four proportions from $6=ax$, using x as a mean.
3. Form proportions from each of the following equations:
   (1) $a^2-b^2=cd.$ (3) $ax+bx+cx=nd+nh+nk.$
   (2) $x^2-1=a^2-b^2.$ (4) $3x^2-11xy-4y^2=4x^2-y^2.$

403. Theorem. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$

To prove $\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$

Proof. Let $\frac{a}{b}=r.$ Then $\frac{c}{d}=r$, $\frac{e}{f}=r$, $\frac{g}{h}=r.$

And $a=br$, $c=dr$, $e=fr$, $g=hr.$ Why?

$a+c+e+g=(b+d+f+h)r.$ Why?

$\frac{a+c+e+g}{b+d+f+h}=r.$ Why?

$\therefore \frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}.$

ILLUSTRATION. $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}.$ Then $\frac{2+4+6}{3+6+9}=\frac{12}{18}=\frac{2}{3}=\frac{4}{6}=\frac{6}{9}.$

THEOREMS OF TRANSFORMATION

404. Theorem. *If four numbers form a proportion, they are in proportion by* ***inversion;*** *that is, the second term is to the first as the fourth is to the third.*

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $\frac{b}{a}=\frac{d}{c}$.

Proof.

$ad=bc$. § 398

Making b and c the extremes, a and d the means, $\frac{b}{a}=\frac{d}{c}$. § 402

405. Theorem. *If four numbers form a proportion, they are in proportion by* ***alternation;*** *that is, the first term is to the third as the second is to the fourth.*

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $\frac{a}{c}=\frac{b}{d}$.

Proof.

$ad=bc$. § 398

$\therefore \frac{a}{c}=\frac{b}{d}$. § 402

406. Theorem. *If four numbers form a proportion, they are in proportion by* ***addition;*** *that is, the sum of the first two terms is to the second as the sum of the last two terms is to the fourth.*

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $\frac{a+b}{b}=\frac{c+d}{d}$.

Proof.

$\frac{a}{b}+1=\frac{c}{d}+1$. § 105

$\therefore \frac{a+b}{b}=\frac{c+d}{d}$. Why?

EXERCISE. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+b}{a}=\frac{c+d}{c}$.

407. Theorem. *If four numbers form a proportion, they are in proportion by* ***subtraction*** *that is, the difference of the first two terms is to the second as the difference of the last two terms is to the fourth.*

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $\frac{a-b}{b}=\frac{c-d}{d}$.

The proof is similar to that of the previous theorem.

EXERCISE. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a-b}{a}=\frac{c-d}{c}$.

408. Theorem. *If four numbers form a proportion, they are in proportion by* ***addition and subtraction;*** *that is, the sum of the first two is to their difference as the sum of the last two is to their difference.*

Given $\frac{a}{b}=\frac{c}{d}$.

To prove $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

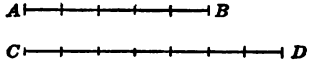
Proof. $\frac{a+b}{b}=\frac{c+d}{d}$. § 406

And $\frac{a-b}{b}=\frac{c-d}{d}$. § 407

Dividing member by member, $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. § 108

409. Remark. The numbers that enter into a proportion may be the numerical measures (§ 332) of geometric magnitudes. In this sense, there may be proportions involving lines and areas.

Thus, if line AB is 5 in. long and CD is 7 in., then $\frac{AB}{CD}=\frac{5}{7}$.



EXERCISES

1. Arrange the following proportions successively by inversion, by alternation, by addition, by subtraction, and by addition and subtraction:

(1) $\frac{8}{15}=\frac{7}{3}$. (2) $\frac{r}{s}=\frac{x}{z}$. (3) $\frac{AB}{CD}=\frac{EF}{GH}$

2. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a-b}{a+b}=\frac{c-d}{c+d}$.

3. If $a-b : b=3 : 7$, find the ratio of a to b.

4. If $a : b=c : d$, prove the following:

(1) $ma : nb=mc : nd$.

(2) $ad : b^2=c : b$.

(3) $ad : b=c : 1$.

(4) $\frac{3a^2+2b^2}{3a^2-2b^2}=\frac{3c^2+2d^2}{3c^2-2d^2}$.

5. Prove that the side of a square is a mean proportional between the sides of a rectangle having the same area as the square.

PROPORTIONAL LINES

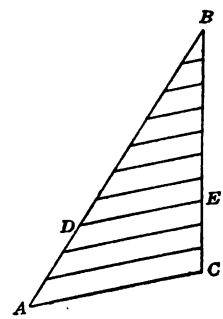
410. Experiment. Construct a triangle ABC. Divide AB into any number of equal parts, as 10. § 322

Through the several points of division on AB draw lines parallel to AC. These lines divide CB into 10 equal parts. § 197

One of these lines, as DE, divides AB and CB so that

$$\frac{AD}{DB}=\frac{3}{7} \text{ and } \frac{CE}{EB}=\frac{3}{7}.$$

$$\therefore \frac{AD}{DB}=\frac{CE}{EB}.$$ Why?



Show by theorems of proportion that:

(1) $\frac{DB}{AD}=\frac{EB}{CE}$. (3) $\frac{AB}{AD}=\frac{CB}{CE}$.

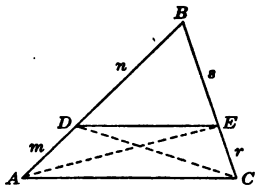
(2) $\frac{AB}{DB}=\frac{CB}{EB}$. (4) $\frac{AD}{CE}=\frac{DB}{EB}$.

The facts shown in this experiment are important because of their frequent use in more advanced mathematics and in practical applications. They are stated as a theorem and proved in the following article.

411. Theorem. ***A straight line parallel to one side of a triangle divides the other two sides in the same ratio.***

Given $\triangle ABC$, having $DE \parallel AC$.

To prove $\frac{m}{n}=\frac{r}{s}$, where $m=AD$, $n=DB$, $r=CE$, and $s=EB$.



Proof. Draw AE and CD.

Since $DE \parallel AC$, $\triangle ADE=\triangle CDE$.
(Having same base and equal altitudes.)

$$\frac{\triangle ADE}{\triangle DBE}=\frac{\triangle CDE}{\triangle DBE}. \qquad \S\ 108$$

Since $\triangle ADE$ and DBE have respectively the bases m and n and the same altitude,

$$\frac{\triangle ADE}{\triangle DBE}=\frac{m}{n}. \qquad \S\ 368$$

Also $$\frac{\triangle CDE}{\triangle BDE}=\frac{r}{s}. \qquad \text{Why?}$$

$$\therefore \frac{m}{n}=\frac{r}{s}. \qquad \S\ 104$$

The foregoing theorem also includes the following obtained from the proportion:

By inversion, $$\frac{n}{m}=\frac{s}{r}.$$

By alternation, $$\frac{m}{r}=\frac{n}{s}.$$

By addition, $$\frac{AB}{AD}=\frac{CB}{CE}, \text{ or } \frac{AB}{DB}=\frac{CB}{EB}.$$

Thus it is seen that: *If a line is drawn parallel to the base of a triangle, either side is to one of its parts as the other side is to its corresponding part.*

EXERCISES

1. In the figure of § 411, compute the parts not given if:

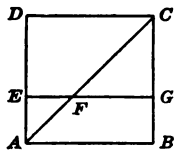
(1) $m=6, n=10, r=8$; (3) $r=24, BC=90, AB=120$;

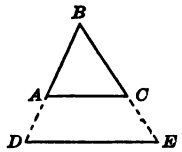
(2) $AB=204, n=45, s=72$; (4) $n=3m, AB+BC=240, r=20$.

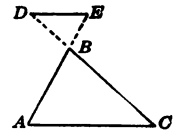
2. Using the values of Exercise 1 (1), find the ratio of $\triangle ACE$ to $\triangle AEB$. Of $\triangle CAB$ to $\triangle CDB$.

3. The straight line drawn through the middle point of one side of a triangle and parallel to a second side bisects the third side.

4. In the square $ABCD$, EG is parallel to AB and intersects AC in F. If $AB=18$ in. and $AE=8$ in., find AF.

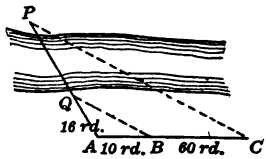






5. Prove the theorem of § 411 if the line parallel to a side is drawn so that it meets the sides produced beyond the base or beyond the vertex, as in the figures.

6. To find the distance between P and Q, two points on opposite banks of a stream, the measurements were made as indicated.

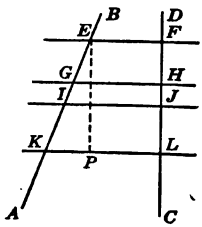


412. Theorem. *If two straight lines are cut by a series of parallel straight lines, the corresponding segments are proportional.*

Given AB and CD, the lines, and EF, GH, IJ, and KL, the series of parallels.

To prove $\dfrac{GE}{HF}=\dfrac{IG}{JH}=\dfrac{KI}{LJ}$.

Draw $EP \parallel FL$. Give proof in full.



EXERCISES

1. Given the segments KI, IG, and GE respectively equal to 5, 3, and 4 in.; find the lengths of the segments of CD if $JH=14$ in.

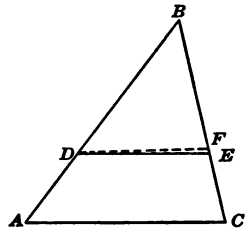
2. Given $LJ : JH : HF = 7 : 3 : 5$, and $KE=45$ in.; find values of KI, IG, and GE.

413. Theorem. *A line which divides two sides of a triangle in the same ratio, is parallel to the third side.*

Given $\triangle ABC$, having DE drawn so that $\frac{AD}{DB}=\frac{CE}{EB}$.

To prove $DE \parallel AC$.

Proof. Suppose that DF is parallel to AC and intersects CB at F.



Then	$\frac{AD}{DB}=\frac{CF}{FB}$.	§ 411
But	$\frac{AD}{DB}=\frac{CE}{EB}$.	Given
Hence	$\frac{CE}{EB}=\frac{CF}{FB}$.	Why?
And	$\frac{CB}{EB}=\frac{CB}{FB}$.	§ 406
Then	$EB=FB$.	§ 400

Hence E coincides with F, and DE coincides with DF. Why?

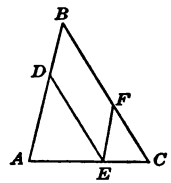
But $DF \parallel AC$, by supposition.

$\therefore DE \parallel AC$.

EXERCISES

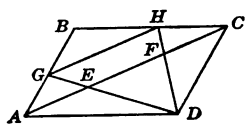
1. $ABCD$ is a quadrilateral having AB parallel to DC. E and F are points in AD and BC respectively such that $AE : ED = BF : FC$. Prove that EF is parallel to AB or DC.

2. Given $\triangle ABC$, D a point in side AB, DE drawn parallel to BC meeting AC in E, and EF drawn parallel to AB meeting BC in F. Prove that $AD : DB = BF : FC$.

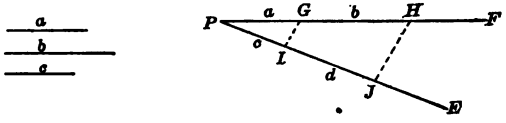


3. From a given point P in the side AB of a triangle ABC draw a straight line to meet AC produced at Q, so that PQ is bisected by BC.

4. $ABCD$ is any parallelogram and AC a diagonal. AE is taken equal to CF. DE and DF are drawn and produced to meet AB and BC respectively, in G and H. Prove that GH is parallel to AC.



414. Problem. *To construct the fourth proportional to three given lines.*



Given three lines a, b, and c.

To construct d such that $\frac{a}{b}=\frac{c}{d}$.

Construction. From any point P draw two lines, PE and PF, making any convenient angle.

Lay off $PG=a$, $GH=b$, and $PI=c$.

Draw GI, and draw $HJ \parallel GI$, meeting PE at J.

Then d is the required fourth proportional.

Proof. In $\triangle PHJ$, $GI \parallel HJ$,

Then $$\frac{PG}{GH}=\frac{PI}{IJ}.$$ Why?

That is, $$\frac{a}{b}=\frac{c}{d}.$$

$\therefore$ d is the required fourth proportional.

EXERCISES

1. Find the fourth proportional to three lines that are $1\frac{1}{2}$ in., $1\frac{3}{4}$ in., and $1\frac{1}{4}$ in. long, respectively. Solve the exercise both by geometry and by algebra and compare the results.

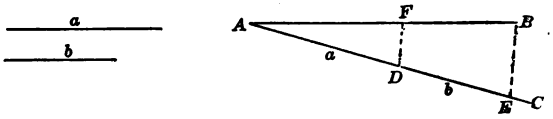
2. Find the value of x in $15 : x = 25 : 20$. Use both algebra and geometry and compare results.

3. Give algebraic and geometric solution for x, if $\frac{36}{50}=\frac{50}{x}$.

4. A given rectangle has a base b and an altitude a. Construct the base of an equivalent rectangle that has a given altitude c.

5. Construct the altitude of a rectangle of given base b, that is equivalent to a square of given side a.

415. Problem. *To divide a given straight line into parts that are in a given ratio.*



Given lines AB, a, and b.

Required to divide AB in the ratio of $\frac{a}{b}$.

Construction. Draw AC making a convenient angle with AB.

On AC lay off $AD=a$ and $DE=b$.

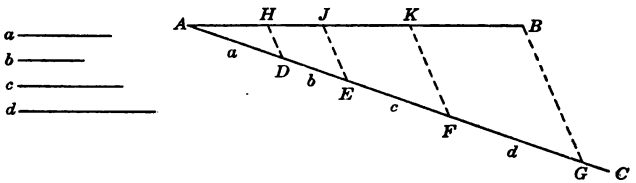
Draw EB and through D draw $DF \parallel EB$.

Then AB is divided in the required ratio by F.

Proof. In $\triangle ABE$, $DF \parallel EB$.

$$\therefore \frac{AF}{FB}=\frac{a}{b}.$$ Why?

416. Problem. *To divide a given straight line into parts proportional to any number of given lines.*



Given the lines AB, a, b, c, and d.

Required to divide AB into parts proportional to a, b, c, and d.

Construction. Draw AC making any convenient angle with AB.

On AC lay off successively a, b, c, and d as shown.

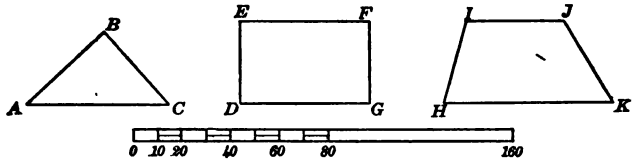
Draw GB and draw lines parallel to GB through F, E, and D.

Then AB is divided as required. Why?

EXERCISES

1. Divide a line 4 inches long into parts that are in the ratio of 7 : 9. Solve both by algebra and by geometry and compare results.

2. In drawing to scale, lines are divided proportionally. Find the length of each side of these figures on a scale of 80 miles to the inch.



3. Construct a scale showing 60 miles to the inch.

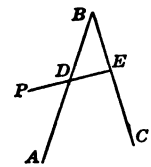
4. Construct a scale showing 50 miles to the centimeter.

5. Divide a line 28 ft. in length into parts that are in the ratio of 1 : 2 : 3 : 4. Solve both by algebra and by geometry and compare results.

6. By a line through a vertex, divide a triangle into parts that are in the ratio of two given straight lines. Into parts that are in the ratio of any number of given lines. § 368

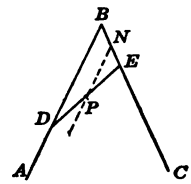
7. Construct a triangle having its sides in the ratio of 2 : 3 : 4, and its perimeter equal to a given straight line.

8. Through a point P without an angle ABC, draw a line intersecting AB in D and CB in E, so that $PD = DE$. So that $PD : DE$ as any given ratio.



9. Through a point P within an angle ABC, draw a line intersecting AB in D and CB in E, so that $DP = PE$. So that $DP : PE$ as any given ratio.

SUGGESTION. Draw $PN \parallel AB$ and lay off $NE = BN$. Could PN be drawn parallel to BC as well?



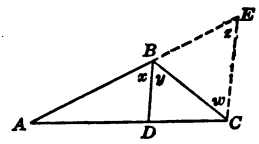
10. ABC is a triangle, and D and E are points in AB and AC, respectively, such that DE is parallel to BC. BE and CD meet at F. Prove that $\triangle ADF$ is equivalent to $\triangle AEF$.

***11.** Divide a line 12 in. long into segments that are in the ratio of $\sqrt{2} : \sqrt{3}$. Construct and compute. Which is the better method? Upon what does the inaccuracy in each depend?

417. Theorem. *The bisector of an angle of a triangle divides the opposite side into two parts which are proportional to the adjacent sides.*

Given $\triangle ABC$, with BD bisecting $\angle ABC$.

To prove $\frac{AD}{DC}=\frac{AB}{BC}$.



Proof. Through C draw $CE \parallel DB$ to meet AB produced at E.

$\angle x=\angle y$, $\angle x=\angle z$, and $\angle y=\angle w$. Why?

Then $\angle z=\angle w$. Why?

And $BE=BC$ Why?

In $\triangle AEC$, since $CE \parallel DB$,

Then $\frac{AD}{DC}=\frac{AB}{BE}$. § 411

$\therefore \frac{AD}{DC}=\frac{AB}{BC}$. Why?

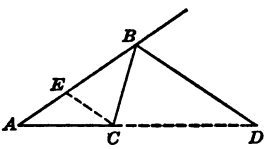
EXERCISES

1. In the figure of § 417:
(1) Given $AB=20$, $AD=14$, $DC=12$; find BC.
(2) Given $AC=34$, $DC=14$, $BC=22$; find AB.

2. The three sides of a triangle are respectively 10, 14, and 22. Find the segments of the sides made by the bisectors of each of the angles.

3. If a straight line bisects both the vertex angle and the base of a triangle, the triangle is isosceles.

4. State and prove the converse of the theorem of § 417.



5. In $\triangle ABC$, BD bisects the exterior angle at B. Show that $AD : CD = AB : BC$.

SUGGESTION. Draw $CE \parallel DB$ and prove that $CB=EB$.

SIMILARITY

418. Geometric figures may be *equivalent*, *similar*, or *equal*. Equivalent figures have the same size, similar figures have the same shape, and equal figures have the same size and shape.*

Similarity is of great practical importance. By similarity, many computations are shortened to a marked degree. Drawings, in general, represent objects of a size different from that of the drawing, but are made so that the reduction or the enlargement is in the same ratio throughout.

An architect in making a plan of a building, constructs the drawing to some scale. A map of a portion of the earth is always drawn to some scale. For instance, if the scale is 1 : 1,000,000, each line on the surface of the earth is 1,000,000 times as long as it is on the map.

419. Definitions. Two polygons are said to be **mutually equiangular** if the angles of one polygon, taken in order, are equal respectively to the angles of the other polygon, taken in the same order.

The pairs of equal angles are called **corresponding angles** of the two polygons.

Two lines, one in each polygon, having the same relative positions with respect to corresponding angles are called **corresponding lines.**

EXERCISES

1. Make a drawing of your school room on a scale of 1 : 96, that is, $\frac{1}{8}$ in. to 1 ft.

2. If a polygon of cardboard is held parallel to a wall and between a light and the wall, what will be the shape of the shadow on the wall?

3. Are two circles of the same shape? Two spheres? Two rectangles? Two triangles? Two equilateral triangles? Two isosceles triangles? Two squares?

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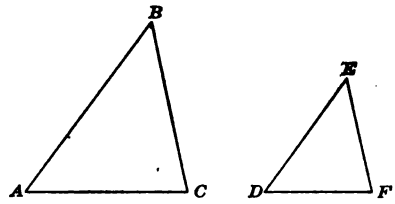
*These statements are not to be considered as definitions. Equal is here used in the sense of congruent.

SIMILAR TRIANGLES

420. Definition. Two triangles that are mutually equiangular are **similar.**



Since $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$, triangle ABC is similar to triangle DEF. This is written $\triangle ABC \backsim \triangle DEF$.*

In the similar triangles ABC and DEF, the equal angles are *corresponding angles* and the sides opposite the equal angles are *corresponding sides.*

421. Theorem. *Two triangles similar to the same triangle are similar to each other.* § 420

422. Theorem. ***Two triangles are similar if two angles of one are equal respectively to two angles of the other.*** § 32

423. Theorem. *Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.*

424. Theorem. *Two isosceles triangles are similar if the angle at the vertex, or a base angle, of one equals the corresponding angle of the other.*

425. Theorem. *Two equilateral triangles are similar.*

426. Theorem. *If the corresponding angles of two triangles have their sides respectively parallel, right side to right side and left side to left side, the triangles are similar.* § 138

427. Theorem. *If the corresponding angles of two triangles have their sides respectively perpendicular, right side to right side and left side to left side, the triangles are similar.* § 140

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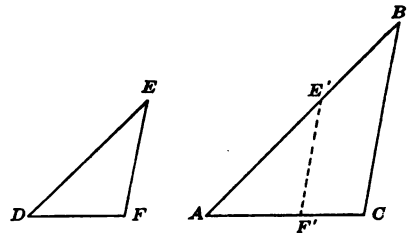
*The symbol $\backsim$ is the letter *s* lying on its side.

428. Theorem. ***The corresponding sides of similar triangles are proportional.***

Given $\triangle ABC \sim \triangle DEF$, with $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.



To prove $\frac{AB}{DE} = \frac{AC}{DF} = \frac{CB}{FE}$.

Proof. Apply $\triangle DEF$ to $\triangle ABC$ so that $\angle D$ coincides with $\angle A$, $\triangle DEF$ taking the position $AE'F'$.

Then $\angle AE'F' = \angle B$, Why?

and $E'F' \parallel BC$. Why?

$\therefore \frac{AB}{AE'} = \frac{AC}{AF'}$. § 411

Or $\frac{AB}{DE} = \frac{AC}{DF}$. Why?

In like manner, by applying $\triangle DEF$ to $\triangle ABC$ so that $\angle F$ and $\angle C$ coincide, it may be proved that

$$\frac{AC}{DF} = \frac{CB}{FE}.$$

$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{CB}{FE}$. Why?

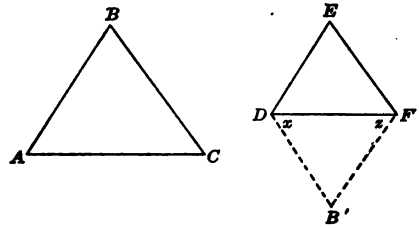
429. Definition. The ratio of two corresponding sides of similar triangles is called the **ratio of similitude.**

EXERCISES

1. Given $\triangle ABC \sim \triangle DEF$, and $AB = 16$, $BC = 14$, $AC = 22$; find the sides of $\triangle DEF$, if $EF = 35$.

2. Given $\triangle ABC \sim \triangle DEF$ and the perimeter of $\triangle ABC = 150$. If $DE = 3$, $EF = 7$, and $DF = 5$, find the sides of $\triangle ABC$.

430. Theorem. ***If two triangles have their corresponding sides proportional, they are similar.***



Given $\triangle ABC$ and DEF, with $\frac{AB}{DE}=\frac{AC}{DF}=\frac{BC}{EF}$

To prove that $\triangle ABC \sim \triangle DEF$.

Proof. At D construct $\angle x=\angle A$. At F construct $\angle z=\angle C$.

Then	$\triangle ABC \sim \triangle DB'F$.	§ 422
And	$\frac{AB}{DB'}=\frac{AC}{DF}$.	§ 428
But	$\frac{AB}{DE}=\frac{AC}{DF}$.	Given
Then	$\frac{AB}{DB'}=\frac{AB}{DE}$.	Why?
	$\therefore DB'=DE$.	§ 400

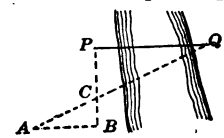
In like manner, it can be proved that $B'F=EF$.

Then	$\triangle DEF=\triangle DB'F$.	§ 27
	$\therefore \triangle ABC \sim \triangle DEF$.	§ 111

EXERCISES

1. Construct a triangle whose sides are respectively 3, 4, and 6 centimeters. Construct a similar triangle, the ratio between the corresponding sides being 1 : 2.

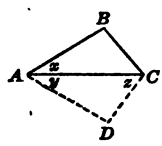
2. Explain method and give proof for measuring the distance from a point P to an inaccessible point Q.

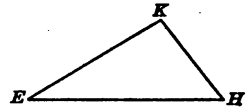


SUGGESTION. Draw $PB \perp PQ$ and $AB \parallel PQ$. Show that $\triangle BAC \sim \triangle PQC$. What measurements are necessary?

431. Theorem. ***If two triangles have an angle of one equal to an angle of the other and the including sides proportional, they are similar.***

Given $\triangle ABC$ and EKH having $\angle x = \angle E$ and $\frac{AB}{EK} = \frac{AC}{EH}$.





To prove that $\triangle ABC \sim \triangle EKH$.

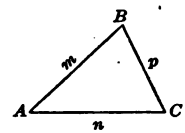
Proof. At A construct $\angle y = \angle E$, and at C construct $\angle z = \angle H$.

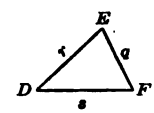
Then	$\triangle ADC \sim \triangle EKH$.	Why?
And	$\frac{AD}{EK} = \frac{AC}{EH}$.	Why?
But	$\frac{AB}{EK} = \frac{AC}{EH}$.	Given
Then	$\frac{AD}{EK} = \frac{AB}{EK}$.	Why?
And	$AD = AB$.	Why?
Then	$\triangle ABC = \triangle ADC$.	Why?
	$\therefore \triangle ABC \sim \triangle EKH$.	§ 111

432. Theorem. ***The areas of two similar triangles are in the same ratio as the squares of any two corresponding sides.***

Given $\triangle ABC \sim \triangle DEF$ with sides m, n, p and r, s, q.

To prove $\frac{\triangle ABC}{\triangle DEF} = \frac{m^2}{r^2}$.





Proof.

	$\frac{\triangle ABC}{\triangle DEF} = \frac{mn}{rs} = \frac{m}{r} \cdot \frac{n}{s}$.	§ 375
But	$\frac{m}{r} = \frac{n}{s}$.	Why?
	$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{m}{r} \cdot \frac{m}{r} = \frac{m^2}{r^2}$.	Why?

433. Summary of Conditions for Similar Triangles. Two triangles are similar if:

(1) Their corresponding angles are equal.

(2) An angle of one equals an angle of the other and the including sides are proportional.

(3) The corresponding sides are proportional.

(4) They are right triangles with an acute angle of one equal to an acute angle of the other, or with any two sides of one proportional to the corresponding sides of the other.

(5) They are isosceles with vertex angles equal, or base angles equal.

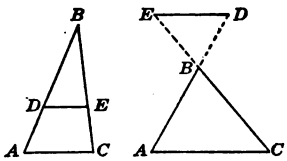
(6) They are equilateral triangles.

(7) Their sides are respectively parallel.

(8) Their sides are respectively perpendicular.

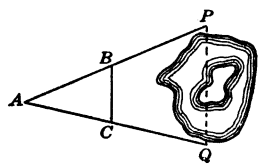
EXERCISES

1. Construct two triangles that are similar but not equivalent. Two that are equivalent but not similar.



2. A straight line parallel to one side of a triangle forms with the other two sides a second triangle similar to the first triangle. In each figure prove that $\triangle ABC \sim \triangle DBE$.

3. The distance PQ may be measured as follows: From a convenient point A, measure the distances AP and AQ. On these lines locate B and C so that $AB : AC = AP : AQ$. Give the further measurements that are necessary and show how the computation is made.



4. Prove that the diagonals of any trapezoid divide each other proportionally.

5. The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it. (Prove by methods of this chapter.)

6. The areas of two similar triangles are in the ratio of 81 : 242; find the ratio of two corresponding sides.

7. On the altitude of an equilateral triangle a second equilateral triangle is constructed. Find the ratio of the areas of the two triangles.

8. If two triangles are similar, altitudes from corresponding vertices are in the same ratio as the corresponding sides. If these altitudes are within the triangles they form two corresponding pairs of similar right triangles.

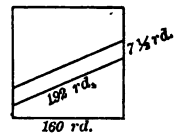
9. If two triangles are similar, the bisectors of corresponding angles, terminated by the opposite sides, are in the same ratio as the corresponding sides.

10. In similar triangles, medians drawn from corresponding vertices are proportional to the corresponding sides.

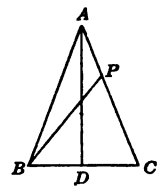
11. The areas of two similar triangles are in the same ratio as:
(1) The squares of corresponding altitudes.
(2) The squares of corresponding medians.
(3) The squares of corresponding bisectors.

12. A median drawn to the base of a triangle bisects all straight lines drawn parallel to the base and terminated by the sides, or the sides produced.

13. A square farm 160 rd. on a side is intersected by a railroad cutting out a parallelogram with sides 192 rd. and $7\frac{1}{2}$ rd. What is the width and the area of the railroad?



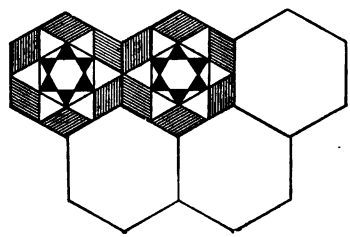
14. In a triangle ABC, $AB = 13$ cm., $BC = 5$ cm., and $AC = 12$ cm. The triangle is folded so that the point A falls on the point B. Find the length of the crease.



15. Triangle ABC is isosceles, having $AB = AC$. $AD \perp BC$. P is a point in AC such that $AP : AB = 1 : 3$. Prove that BP bisects AD.

16. A triangle PQR has PQ and QR divided at D and E respectively so that $PD : DQ = QE : ER = 1 : 3$. PE and RD intersect at O. Find the following ratios:
$\triangle PDO : \triangle OPR : \triangle OER : \triangle PQR$.

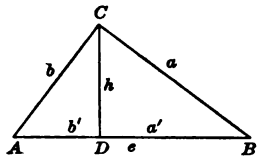
17. The figure is a design from a linoleum pattern. Find the ratios of the black, the white, and the shaded portions.



434. Theorem. *In any right triangle, if the altitude is drawn from the vertex of the right angle to the hypotenuse:*

(1) *The two triangles thus formed are similar to the given triangle and to each other.*

(2) *The altitude is the mean proportional between the segments of the hypotenuse.*



Given rt. $\triangle ACB$, having the altitude CD from the vertex of rt. $\angle C$.

To prove (1) $\triangle ACB \sim \triangle ADC \sim \triangle CDB$, and (2) $b' : h = h : a'$.

Proof. In $\triangle ADC$ and ACB, $\angle A$ is common.

Then	$\triangle ADC \sim \triangle ACB$.	§ 423
	$\triangle CDB \sim \triangle ACB$.	Why?
	$\triangle ADC \sim \triangle CDB$.	§ 421
	$\therefore b' : h = h : a'$.	§ 428

In the following four theorems use the notation of § 434.

435. Theorem. *Each side about the right angle is the mean proportional between the hypotenuse and its adjacent segment;* that is, $a' : a = a : c$, and $b' : b = b : c$.

436. Theorem. *The two segments of the hypotenuse and the squares of the two sides are proportionals;* that is, $a' : b' = a^2 : b^2$.

SUGGESTION. In § 435 take the product of the means equal to the product of the extremes, and divide member by member.

437. Theorem. *The square of the hypotenuse is to the square of either side as the hypotenuse is to the segment of the hypotenuse adjacent to that side;* that is, $c^2 : a^2 = c : a'$, and $c^2 : b^2 = c : b'$.

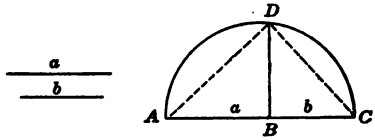
438. Theorem. *The square of the hypotenuse equals the sum of the squares of the two sides.*

Proof. In each proportion of § 435, take the product of the means equal to the product of the extremes.

Then	$a^2 = a'c$ and $b^2 = b'c$.
Adding,	$a^2 + b^2 = (a' + b')c$.
But	$a' + b' = c$.
	$\therefore c^2 = a^2 + b^2$.

439. Theorem. *A perpendicular drawn to a diameter from any point on the circle is a mean proportional between the segments of the diameter.* § 434

440. Problem. ***To construct the mean proportional between two given lines.***



Given lines a and b.

Required to construct the mean proportional between a and b.

Construction. Draw the straight line AC, making $AB=a$ and $BC=b$.

Upon AC as a diameter describe a semicircle.

At B erect a perpendicular to meet the semicircle at D.

Then BD is the mean proportional between a and b. § 439

REMARK. This problem gives another method for constructing a square equivalent to a given rectangle; for a side of the square is the mean proportional between the sides of the rectangle.

EXERCISES

1. Construct a square equivalent to a rectangle 3 cm. by 2 cm.

SUGGESTION. Use § 440. Also use § 390.

2. Construct a line equal to $\sqrt{21}$ in length.

SUGGESTION. $21=3\times7$.

3. Construct a rectangle equivalent to a given square and having its perimeter equal to a given straight line. When will the construction be impossible?

4. If the altitude of a right triangle is three times the base, in what ratio does the perpendicular from the right angle divide the hypotenuse?

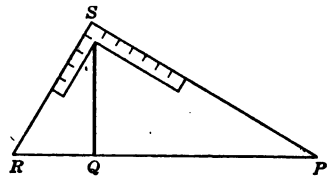
5. In the following, find the lengths omitted. Use figure of § 434.

No.	AB	AC	BC	DC	AD	DB
(1)	13	12				
(2)				12	$3\frac{1}{2}$	
(3)					16	4
(4)		20		12		
(5)		30				50

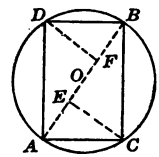
6. Given the sum of two lines and the mean proportional between them, to construct the two lines.

7. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided by the smaller circle into parts that are in the same ratio.

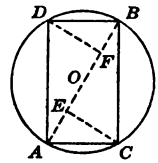
***8.** Show how the distance from the point Q to an inaccessible point P can be determined by placing a carpenter's square on top of a pole set at Q as shown in the figure.



***9.** The strongest rectangular beam that can be cut from a round log is one in which the breadth is to the depth as $1 : \sqrt{2}$. It can be found graphically as follows: Draw the diameter AB, and trisect it at E and F. Draw perpendiculars to the diameter at E and F to meet the circumference at C and D respectively. Then $ACBD$ is rectangular and of the proper dimensions. Prove. § 436



***10.** The rectangular beam of greatest stiffness that can be cut from a round log is one in which the breadth is to the depth as $1 : \sqrt{3}$. It can be found graphically as follows: Draw the diameter AB, and divide it into four equal parts by points E, O, and F. Draw perpendiculars to the diameter at E and F to meet the circumference at C and D respectively. Then $ACBD$ is rectangular and of the proper dimensions. Prove. § 436



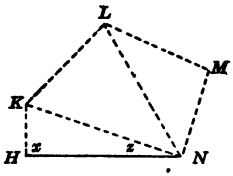
***11.** What are the dimensions of the strongest rectangular beam that can be cut from a log having a diameter of 24 inches?

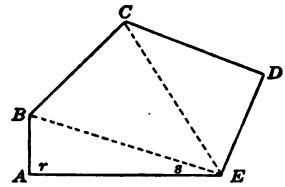
Ans. 19.6 in. by 13.9 in.

SIMILAR POLYGONS

441. Two polygons composed of the same number of triangles, similar each to each and similarly placed, are **similar polygons.**

442. Problem. *Construct a polygon similar to a given polygon having given a side corresponding to a side of the given polygon.*





Given polygon $ABCDE$ and side HN corresponding to AE.

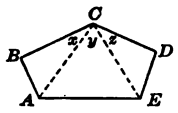
Required to construct a polygon similar to $ABCDE$.

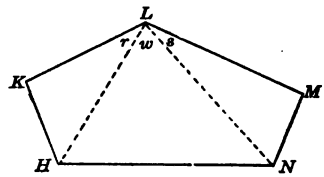
Construction. Draw the diagonals EB and EC. Construct $\angle x = \angle r$, and $\angle z = \angle s$. Then $\triangle HKN \backsim \triangle ABE$. Why?

Construct $\triangle KLN$ on KN similar to $\triangle BCE$, and $\triangle LMN$ on LN similar to $\triangle CDE$.

Then $HKLMN \backsim ABCDE$. § 441

443. Theorem. *The corresponding angles of two similar polygons are equal.*





Given the similar polygons $ABCDE$ and $HKLMN$.

To prove $\angle B = \angle K$, $\angle C = \angle L$, etc.

Proof. Let the polygons be composed of similar triangles as shown.

$\angle B = \angle K$. § 420

Also $\angle x = \angle r$, $\angle y = \angle w$, $\angle z = \angle s$. Why?

Then $\angle x + \angle y + \angle z = \angle r + \angle w + \angle s$. Why?

$\therefore \angle BCD = \angle KLM$.

Similarly, any two corresponding angles may be proved equal.

444. Theorem. *The corresponding sides of two similar polygons are proportional.*

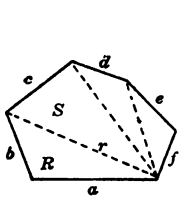


FIG. 1

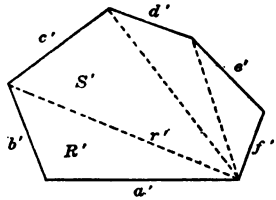


FIG. 2

Given the similar polygons shown in Fig. 1 and Fig. 2.

To prove $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}=\frac{e}{e'}=\frac{f}{f'}$.

Proof. $\triangle R \backsim \triangle R'$. § 441

Then $\frac{a}{a'}=\frac{b}{b'}=\frac{r}{r'}$. § 428

Also since $\triangle S \backsim \triangle S'$, $\frac{c}{c'}=\frac{r}{r'}$. Why?

Then $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$. Why?

In like manner, it may be proved that $\frac{c}{c'}=\frac{d}{d'}$, etc.

$\therefore \frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}=\frac{e}{e'}=\frac{f}{f'}$. Why?

445. Theorem. *The perimeters of two similar polygons are in the same ratio as any two corresponding sides.*

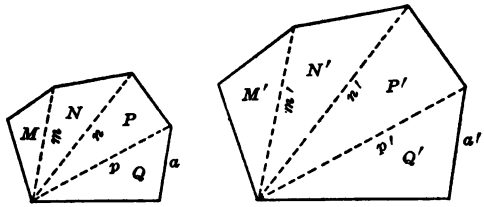
In the similar polygons shown in Fig. 1 and Fig. 2, § 444,

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}=\frac{e}{e'}=\frac{f}{f'}.$$

$$\therefore \frac{a+b+c+d+e+f}{a'+b'+c'+d'+e'+f'}=\frac{a}{a'}=\frac{b}{b'}=\cdots\frac{f}{f'}.$$ § 403

Or, using p and p' for perimeters, $\frac{p}{p'}=\frac{a}{a'}=\frac{b}{b'}=\cdots\frac{f}{f'}$.

446. Theorem. *The areas of two similar polygons are in the same ratio as the squares of any two corresponding sides.*



Given the two similar polygons, having areas S and S', and corresponding sides a and a'.

To prove $$\frac{S}{S'}=\frac{a^2}{a'^2}.$$

Proof. Draw the diagonals dividing the polygons into pairs of similar triangles, M and M', N and N', etc.

Then $$\frac{M}{M'}=\frac{m^2}{m'^2}=\frac{N}{N'}=\frac{n^2}{n'^2}=\frac{P}{P'}=\frac{p^2}{p'^2}=\frac{Q}{Q'}.$$ § 432

$$\therefore \frac{M}{M'}=\frac{N}{N'}=\frac{P}{P'}=\frac{Q}{Q'}.$$ Why?

Now $$\frac{M+N+P+Q}{M'+N'+P'+Q'}=\frac{Q}{Q'}.$$ § 403

That is, $$\frac{S}{S'}=\frac{Q}{Q'}.$$

But $$\frac{Q}{Q'}=\frac{a^2}{a'^2}.$$ § 432

$$\therefore \frac{S}{S'}=\frac{a^2}{a'^2}.$$ Why?

EXERCISES

1. Construct a hexagon similar to a given hexagon, making each side twice the corresponding side of the given hexagon.

2. Draw two hexagons which have the angles of one equal respectively to the angles of the other, but which do not have their corresponding sides proportional.

3. Draw two quadrilaterals which have the sides of one proportional respectively to the sides of the other, but which do not have their corresponding angles equal.

4. Two similar polygons have perimeters in the ratio of 9 : 14. Find the side in the second, corresponding to a side 6 ft. in length in the first.

5. Prove that the perimeters of two similar polygons are in the same ratio as any two corresponding diagonals.

6. The sides of a pentagon are respectively 3, 7, 2, 8, 9. Find the sides of a similar pentagon if the ratio of its perimeter to that of the first is as 3 : 5.

7. The areas of two similar polygons are in the same ratio as the squares of corresponding diagonals.

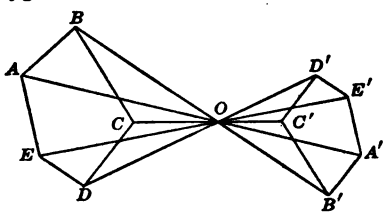
8. The areas of two similar polygons are 225 sq. in. and 625 sq. in.; find the ratio of their perimeters. One side of the first is 8 in.; find the corresponding side in the second.

9. Construct a pentagon similar to a given pentagon and having an area twice as great.

10. Construct a quadrilateral similar to two given similar quadrilaterals and equivalent to their sum.

SUGGESTION. Construct a side such that its square equals the sum of the squares of two corresponding sides in the given quadrilaterals.

11. If, on the three sides of a right triangle, similar polygons are so constructed as to make the sides of the triangle corresponding sides of the similar polygons, the polygon on the hypotenuse is equal to the sum of the other two polygons.

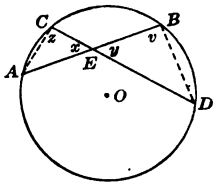


12. Let O be any point within or without polygon $ABCDE$. Draw lines through O and the several vertices of the polygon. From A', any point in OA, draw $A'B' \parallel AB$ to intersect OB in B'. Similarly draw $B'C' \parallel BC$, $C'D' \parallel CD$, and $D'E' \parallel DE$. Prove that $E'A' \parallel EA$, and prove that $A'B'C'D'E' \sim ABCDE$.

NOTE. The point O is called the **center of similitude of the polygons** $ABCDE$ and $A'B'C'D'E'$.

PROPORTIONAL LINES CONNECTED WITH CIRCLES

447. Theorem. *If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.*



Given circle O and the chords AB and CD intersecting at E.

To prove $AE \cdot BE = CE \cdot DE$.

Proof. Draw AC and BD.

$\angle x = \angle y$. Why?

$\angle z = \angle v$. Why?

Then $\triangle ACE \sim \triangle DBE$. § 422

And $\dfrac{AE}{DE} = \dfrac{CE}{BE}$. Why?

$\therefore AE \cdot BE = CE \cdot DE$. Why?

448. Remark. This theorem is often stated as follows:

If two chords intersect within a circle, the rectangle formed by the segments of one equals the rectangle formed by the segments of the other.

EXERCISES

1. In the figure of § 447, $AE = 10$, $EB = 18$, $CE = 8$; find ED.

2. Given a rectangle with sides 5 and 8. Construct an equivalent rectangle having one side 7.

SUGGESTION. Use § 447, making the sides of the rectangles segments of intersecting chords. Also, construct by § 414.

3. Draw a circle with a radius of 5 cm., and locate a point P 4 cm. from the center. Draw a chord through P such that the perpendicular from the center to the chord shall be 2 cm. in length.

4. A point on a diameter divides it into segments of lengths 5 and 20; find the length of the perpendicular drawn to the diameter at the given point and terminated by the circle.

5. A chord 16 in. long is drawn in a circle of radius 10 in. Find the distance from the center of the circle to the chord.

SUGGESTION. Let x = number of inches the chord is from the center.

Then $(10-x)(10+x)=8^2$.

$\therefore x=6$.

6. A chord 16 in. long lies 6 in. from the center of a circle. Find the radius of the circle.

7. In a circle of 5 cm. radius, chords are drawn through a point 2 cm. from the center. Find the product of the segments of any of these chords.

8. The segments of one of two intersecting chords are x and $13-x$, and of the other 4 and 10. Find x and the length of the first chord.

9. The segments of one of two intersecting chords are $2x-1$ and $6x+1$, and of the other 6 and $6\frac{1}{2}$. Find x and the length of the first chord.

10. The segments of one of two intersecting chords are $x+y$ and $x-y$, and of the other 5 and 13. Find x, y, and the length of the first chord if $x-y=4$.

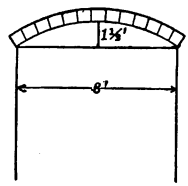
11. Derive the formulas of Exercise 10, page 184, by using § 447.

12. Find the height of a segment whose chord is 12 in. long in a circle having a radius of 8 in.

13. In a circle having a radius of 10 in., find the width of a segment which has a height of 3 in.

14. A 10-foot stick is placed with its ends against the rail at a curve in a street car track. If the center of the stick is 3 in. from the rail, find the radius of the curve in the car track.

15. The arch above a door 8 feet wide has a height of $1\frac{1}{2}$ ft. If the arch is an arc of a circle, find the radius of the circle.

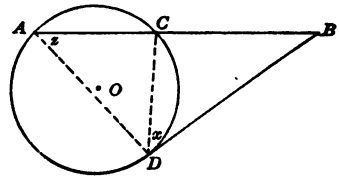


16. In a circle with a radius of 4 cm., a chord DE is drawn 7 cm. in length. F is a point in DE such that $DF=5$ cm. Find the distance of F from the center of the circle.

449. Secants and Tangents. The part of a secant which lies between an external point and the second point of intersection with the circle is the **length of the secant from the external point.**

The part of a tangent which lies between an external point and the point of contact with the circle is the **length of the tangent.**

450. Theorem. *If a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the secant and its external segment.*



Given BD a tangent and BA a secant drawn to the circle O from the external point B.

To prove $\frac{BA}{BD}=\frac{BD}{BC}$.

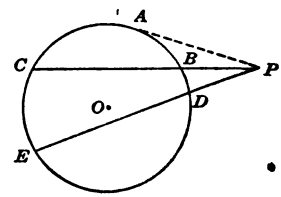
Proof. Draw the chords AD and DC.

In $\triangle ABD$ and DBC,	$\angle B=\angle B$.	Iden.
	$\angle x=^\circ \frac{1}{2}\overset{\frown}{DC}$ and $\angle z=^\circ \frac{1}{2}\overset{\frown}{DC}$.	Why?
Then	$\angle x=\angle z$.	Why?
And	$\triangle ABD \sim \triangle DBC$.	§ 422
	$\therefore \frac{BA}{BD}=\frac{BD}{BC}$.	Why?

451. Theorem. *If a tangent and a secant are drawn to a circle from an external point, the square of the tangent is equal to the product of the secant and its external segment.*

$$\overline{BD}^2=BA\cdot BC. \qquad \text{§ 450}$$

452. Theorem. *If two secants are drawn to a circle from an external point, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*



Given circle O, the external point P, and secants PC and PE.

To prove $PC\cdot PB=PE\cdot PD$.

Proof. Draw the tangent PA.

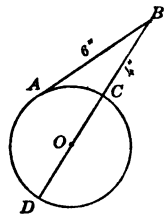
Then	$PC\cdot PB=\overline{PA}^2$.	§ 451
Also	$PE\cdot PD=\overline{PA}^2$.	
	$\therefore PC\cdot PB=PE\cdot PD$.	Why?

EXERCISES

1. A secant from an external point P to a circle cuts the circle in points A and B. If $PA=10$ and $PB=40$, find the length of a tangent from P to the circle.

2. In a circle with a radius 5 cm., a chord CD 6 cm. long is drawn. Produce CD through D to E making $DE=4$ cm. Find the length of the tangent drawn from E to the circle, and the distance from E to the center of the circle.

3. In the figure, CD is a diameter of the circle. Tangent $AB=6$ in. $BC=4$ in. Find the radius of the circle.



4. Find the length of the tangent, the whole secant, and the external segment when given:

	Tangent	Whole Secant	External Segment
(1)	x	$x+5$	$x-3$
(2)	x	$x-4$	$x-12$
(3)	x	$x+16$	$x-2$

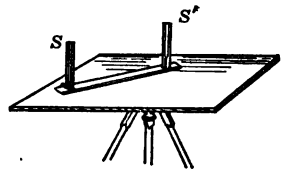
5. If two secants are drawn from a point outside a circle to the circle, find the parts not given in the following table:

No.	First Secant		Second Secant	
	Whole	External	Whole	External
(1)	$26+x$	x	40	14
(2)	$x+15$	15	30	24
(3)	30	13	$10+x$	10
(4)	30	7	$11+x$	x
(5)	20	9	$24+x$	x

6. Draw a circle with a radius of 3 cm., and mark any point P on the circumference. From this point draw a tangent PQ, 4 cm. long. Measure the distance of Q from the center of the circle and check the result.

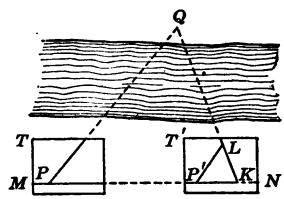
7. Draw a circle with a radius of 5 cm. Draw any straight line through the center, and find a point on this line from which the tangent to the circle will be 5 cm. in length.

453. Plane-Table. The **plane-table** is an instrument used to locate points in making maps, and to find distances. On account of the ease and rapidity with which it can be operated it is much used where only a moderate approach to accuracy is required.



The plane-table consists essentially of a mounted drawing board as shown in the figure. Upon the board is a ruler with two sights S and S'. A telescope often takes the place of the sights.

In using the plane-table, a figure similar to that being measured is drawn on paper placed upon the table. For example, it is required to find the distance from P to Q. (1) Locate a convenient base line MN both on the ground and on the paper. (2) With the table in the position T, sight the ruler toward Q and draw a line along it from P. (3) Move the table to another position T' on the base line, and at K, so chosen that $P'K$ represents PK to some convenient scale, sight the ruler toward Q and draw the line KL, thus forming the triangle $P'LK$. The distance desired can then be computed by similar triangles. The lines on the paper should be measured with the greatest care for a slight error would make a much greater error in the actual distances.

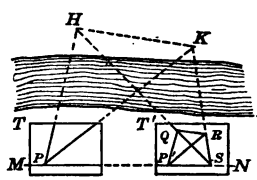


EXERCISES

1. Prove that $\triangle PQK \backsim \triangle P'LK$.

2. If $PK=40$ rd., $P'K=5$ in., and $P'L=7$ in., find PQ.

3. Show how to find the distance between the two inaccessible points H and K by means of the plane-table. (See Exercise 9, page 25.)

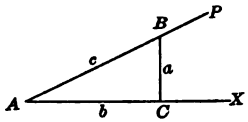


*4. Measure various distances by means of the plane-table and check the results by actual measurement.

SOME TRIGONOMETRIC RATIOS AND THEIR USES

454. Trigonometry is based upon the ratios between the lengths of certain lines connected with an angle. By means of these ratios various useful and interesting problems can be solved.

455. Trigonometric Ratios. Draw any angle XAP. From any point in either side drop a perpendicular upon the other side, as BC. Let $AB=c$, $AC=b$, and $BC=a$.



The ratio $\frac{a}{c}$ is called the **sine** of $\angle A$ (**sin** A).

The ratio $\frac{b}{c}$ is called the **cosine** of $\angle A$ (**cos** A).

The ratio $\frac{a}{b}$ is called the **tangent** of $\angle A$ (**tan** A).

The ratios in the table on page 220 are for angles of any integral number of degrees from 0° to 90°. The values are approximated to three decimal places. The values, however, can be approximated to any desired number of decimals.

EXERCISES

1. Show that the values of the trigonometric ratios are not changed by changing the position of the perpendicular.

2. Show that if the angle is changed the values of the trigonometric ratios are changed.

3. Use a protractor and construct an angle of 10°. Draw a perpendicular, measure the lines a, b, and c of § 455, and determine the trigonometric ratios to two decimal places. Construct angles and determine the ratios for 20°, 33°, 42°, 45°, 67°, 75°, 80°. Compare results with table on page 220.

4. Determine the trigonometric ratios for 45°, 30°, and 60° without constructing the angles and measuring the lines. Are there any other acute angles for which the ratios can be determined in this manner?

(See Exercises 5 and 6, page 166.)

TABLE OF TRIGONOMETRIC RATIOS

Angle	sin	cos	tan	Angle	sin	cos	tan
0°	.000	1.000	.000	45°	.707	.707	1.000
1°	.017	1.000	.017	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.999	.052	48°	.743	.669	1.111
4°	.070	.998	.070	49°	.755	.656	1.150
5°	.087	.996	.087	50°	.766	.643	1.192
6°	.105	.995	.105	51°	.777	.629	1.235
7°	.122	.993	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	55°	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	60°	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	65°	.906	.423	2.145
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	70°	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	75°	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	80°	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.993	.122	8.144
39°	.629	.777	.810	84°	.995	.105	9.514
40°	.643	.766	.839	85°	.996	.087	11.430
41°	.656	.755	.869	86°	.9976	.070	14.301
42°	.669	.743	.900	87°	.9986	.052	19.081
43°	.682	.731	.933	88°	.9994	.035	28.636
44°	.695	.719	.966	89°	.9998	.017	57.290
45°	.707	.707	1.000	90°	1.000	.000	.....

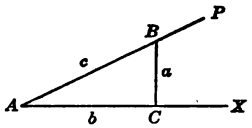
456. Use of the Trigonometric Ratios. One use of the trigonometric ratios is to solve right triangles.

The three lines a, b, and c form a right triangle. Further, each of the three definitions:

$$\sin CAB = \frac{a}{c},$$

$$\cos CAB = \frac{b}{c},$$

$$\tan CAB = \frac{a}{b},$$



is an equation in which three parts of a triangle, two sides and an angle, are involved. If, then, any two of these parts are given, the other can be determined by solving the equation.

For example, if a and c are known, then $\sin CAB$ can be computed and the angle CAB found from the table.

If the angle and a are known, then $c = \frac{a}{\sin CAB}$. If the angle and c are known, then $a = c \sin CAB$.

EXERCISES

1. Solve $\cos CAB = \frac{b}{c}$ and $\tan CAB = \frac{a}{b}$ for each of the parts.

2. To find the height of the tree CB, the distance AC and the angle at A were measured. (Fig. 1.) Using the table on page 220, find CB if $AC = 65$ ft. and $\angle CAB = 68°$.

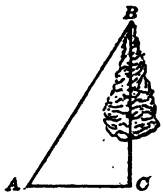


FIG. 1

3. In a right triangle, it is customary to use the letters as shown in Fig. 2. Solve the following right triangles:

(1) Given $A = 27°$, $b = 14$ ft.; find B, a, and c.
(2) Given $A = 63°$, $c = 45$ ft.; find B, a, and b.
(3) Given $A = 52°$, $a = 3$ in.; find B, b, and c.
(4) Given $B = 47°$, $a = 19$ in.; find A, b, and c.
(5) Given $B = 82°$, $c = 102$ ft.; find A, a, and b.
(6) Given $B = 21°$, $b = 27$ in.; find A, a, and c.
(7) Given $a = 27$ ft., $b = 16$ ft.; find A, B, and c.
(8) Given $a = 14$ in., $c = 21$ in.; find A, B, and b.
(9) Given $b = 16$ ft., $c = 40$ ft.; find A, B, and a.

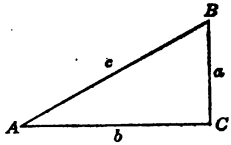
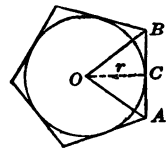


FIG. 2

4. If a vertical shaft 20 ft. high casts a shadow 26 ft. long on level ground, find the angle the sun's rays make with the horizontal.

5. At 60 ft. from the base of a Washington fir tree the angle of elevation of the top is 75°. Find the height of the tree. *Ans.* 224 ft.

6. Find the side of a regular pentagon circumscribed about a circle having a radius of 8 in.



SOLUTION. $\angle AOB = 72°$. Why?

Then $\angle COB = 36°$.

$$\tan COB = \frac{CB}{r}.$$

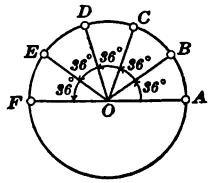
Then $CB = r \tan COB$.

$\therefore CB = 8 \times 0.727 = 5.816$.

And $AB = 2 \times 5.816 = 11.632$.

7. Find the side of a regular pentagon inscribed in a circle having a radius of 20.

8. Locate the centers of the holes B and C of the figure, by finding the distance each is to the right and above the center O. The radius of the circle is 1.5 in.



9. A chord of a circle is 16 ft. long, and the angle which it subtends at the center is 47°. Find the radius of the circle.

10. Find the area of a triangle if two sides are 20 ft. and 30 ft. and the included angle 40°.

11. The radius of a circle is 8 ft. What angle will a chord 12 ft. long subtend at the center?

12. The base of an isosceles triangle is 16 in., and the angle at the vertex is 72° 30′. Find the equal sides and the altitude.

13. The base of an isosceles triangle is 28 ft., and each of the base angles is 48° 15′. Find the equal sides and the altitude.

14. A building 80 ft. long and 60 ft. wide has a roof inclined at an angle of 36° with the horizontal. Find the area of the roof. *Ans.* 5930 sq. ft.

15. The angle of elevation of a balloon from a point due south of it is 60°, and from another point one mile due west of the former point, the angle of elevation is 45°. Find the height of the balloon. (See Exercise 6, p. 55.) *Ans.* 1.225 miles.

16. Find the shorter altitude and the area of a parallelogram whose sides are 10 ft. and 25 ft. respectively, and the angle between the given sides 75°. *Ans.* Alt. 9.66 ft.; area 241.5 sq. ft.

QUESTIONS

1. Define ratio. Define proportion. Give the theorems for transformation of a proportion.

2. Define similar triangles. Similar polygons.

3. Under what conditions are two triangles similar? Two polygons?

4. If the area of a polygon is double that of a similar polygon, how do the sides compare?

5. If the sides of a polygon are double the corresponding sides of a similar polygon, how do the areas compare?

6. If two sides of a triangle are doubled and the third side is left unchanged, how is the area affected?

7. Give two ways for constructing a mean proportional between two lines.

8. Which of the following figures are similar if they are (*a*) mutually equilateral; (*b*) mutually equiangular: triangles, quadrilaterals, octagons?

9. If a perpendicular is drawn in a right triangle from the vertex of the right angle to the hypotenuse, state the conclusions that follow.

***10.** In an isosceles right triangle, what is the ratio of one of the equal sides to the hypotenuse? What is the sin 45°? cos 45°?

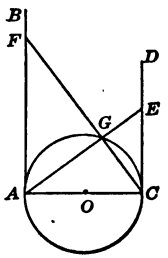
***11.** In an equilateral triangle, what is the ratio of the altitude to a side? What is the sin 60°? cos 30°? tan 60°? tan 30°?

GENERAL EXERCISES

THEOREMS

1. If two circles intersect, their common chord produced bisects their common tangents.

2. In the figure, AB and CD are tangents at the extremities of a diameter. AE and CF are drawn to intersect at G on the circle. Prove that AC is a mean proportional between AF and CE.



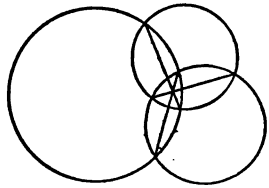
3. Given $ABCD$ a parallelogram. Any straight line AEF is drawn cutting BC in E and DC produced in F. Show that half the parallelogram is a mean proportional between triangles ABE and ADF.

4. If two chords of a circle intersect at right angles, the sum of the squares of their segments is equal to the square of the diameter.

5. The four triangles into which a quadrilateral is divided by its diagonals are proportional.

6. Given ABC any triangle, D a point in AB such that $AD = 3DB$, and E a point on AC such that $CE = 3EA$. Show that area of $\triangle CDE$ is 9 times $\triangle BDE$.

7. If two circles intersect, the tangents drawn to them from any point in their common chord produced are equal to each other.

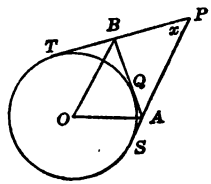


8. If three circles whose centers are not in the same straight line are drawn so that each intersects the other two, the common chords of each pair meet in a point.

9. If the sides of a right triangle are in the ratio of 3 : 2, prove that the perpendicular from the vertex of the right angle to the hypotenuse divides it into parts in the ratio of 9 : 4.

10. If a square is inscribed in a right triangle so that one of its sides is on the hypotenuse, show that the three segments of the hypotenuse are in a proportion.

11. The tangents PS and PT are drawn from the point P to the circle O. A third tangent is drawn at any point Q in the minor arc TS and intersects the first two tangents in the points A and B respectively. If x stands for the angle SPT, show that $\angle AOB = 90° - \frac{1}{2}x$. Also if Q is on the major arc, show that $\angle AOB = 90° + \frac{1}{2}x$.



CONSTRUCTIONS

1. Draw a parallelogram with sides 10 cm. and 7 cm. and included angle 60°. Draw a rhombus having each side 10 cm. and equal in area to the parallelogram.

2. Divide a given straight line into two parts such that the square on one part is double the square on the other part.

SUGGESTION. Divide the line in the ratio of the side of a square to its diagonal.

3. Construct a square equivalent to $\frac{3}{5}$ of a given square.

4. Draw a square that has the sum of a side and a diagonal equal to 5 inches.

5. Draw a square in which the difference between a diagonal and a side is 1 inch.

6. Bisect a given triangle by a straight line drawn parallel to one of its sides.

7. Cut off from a given triangle, a part equal to one-third of its area by a straight line drawn parallel to one of its sides.

8. Trisect a given triangle by straight lines parallel to one of its sides.

9. Divide a given straight line into two parts such that the areas of the squares constructed upon them shall be in a given ratio.

SUGGESTION. Use § 436.

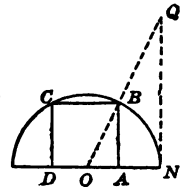
10. Construct a rectangle having an area of 144 sq. in. and a perimeter of 50 in. Solve by algebra and compare the results.

SUGGESTION. Construct a semicircle with a diameter of 25 in., and a line 12 in. long which meets the circle and is perpendicular to the diameter.

11. Construct a rectangle equivalent to a given triangle and having one of its sides equal to a given straight line.

12. Inscribe a square in a given semicircle.

SUGGESTION. Evidently OA must be $\frac{1}{2}AB$. It is then necessary to determine where to draw AB. This can be done by drawing $NQ \perp ON$ and equal to 2 ON. Then B is determined by connecting Q and O.



13. Draw a circle with a radius of 33 mm. Locate any point P outside the circle, and draw a straight line from P such that the chord cut off upon it by the circle shall be 4 cm. in length.

SUGGESTION. With the same center, construct a circle to which the chord 4 cm. long is tangent.

14. The sides of a triangle are 4, 2, and 3, and the longest side of a similar triangle is 10. Construct the second triangle. Compute the length of its sides and compare.

15. Circles are described passing through two fixed points. Find the locus of a point from which the tangents to all the circles are equal.

***16.** The radius of a fixed circle is 28 mm., and a chord AB of the circle is divided at P so that $AP \times PB = 225$ sq. mm. Construct the locus of P.

COMPUTATIONS

1. How far from the eye should a silver dollar, which has a diameter of $1\frac{1}{2}$ in., be held to cover the moon's disk if the moon is 240,000 miles distant and 2160 miles in diameter?

2. If P is a point at a distance of 7 cm. from the center of a circle, and PQS a secant such that $PQ = 5$ cm. and $QS = 3$ cm., find the radius.

3. In a circle of radius 4 in., a chord DE, 7 in. long is drawn. If P is a point in DE such that $DP = 5$ in., find the distance of P from the center.

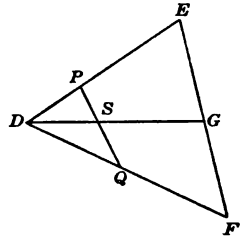
4. Given $ABCD$ a parallelogram, and P a point in AB such that $AP : PB = m : n$. What fraction of the parallelogram is the triangle APD?

5. Given $ABCD$ a parallelogram, and P a point in the diagonal BD such that $BP : PD = 2 : 5$. What fraction of the parallelogram is triangle ABP?

6. Given $ABCD$ a parallelogram, and P a point in the diagonal BD produced, such that $BP : PD = 10 : 7$. What fraction of the parallelogram is the triangle PAD?

7. The altitude of a triangle is 10 ft. At what distances from the vertex should lines parallel to the base be drawn to divide the triangle into three equal parts?

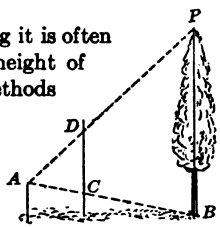
8. Given DEF, a triangle, P and Q points in DE and DF such that $DP : PE = 3 : 5$, and $DQ : QF = 7 : 8$. In what ratio is PQ cut by the median DG?



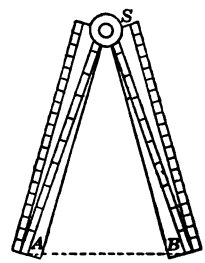
MISCELLANEOUS APPLICATIONS

1. A pair of *proportional dividers* are shown in the figure. They are made of two graduated bars, AB and CD, having slots such that the adjusting screw S can be so placed that the openings AC and DB may have a given ratio. Show how they may be used in constructing similar figures.

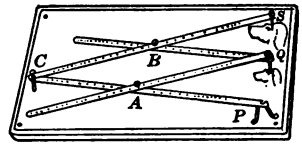
2. In practical work of timber cruising it is often necessary to determine the approximate height of a standing tree. One of the simplest methods is to measure the shadow of the tree and the shadow of a straight pole of known length, then by similar triangles to find the height of the tree. A method used when the sun is not shining is to set two poles in line with the tree as shown in the figure. By measuring AB, AC, and CD, the height BP can be determined. Show how this can be done.



3. The instrument shown in the figure is a *sector*. It is used to divide a line into any number of equal parts. The two arms are jointed at S. To use it, open so that the distance from A to B equals the length of the line. Then the transverse distance, as $\frac{1}{5}$ the distance down from S, equals $\frac{1}{5}$ the length of the line. Prove this.



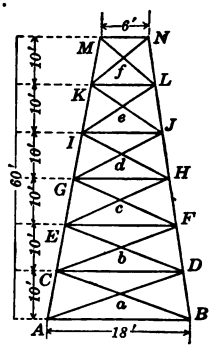
4. The instrument shown here is a *pantograph*. It is used to draw a plane figure similar to a given plane figure. Thus maps and drawings may be enlarged or reduced. The four bars are parallel in pairs and adjustable at the points A and B.



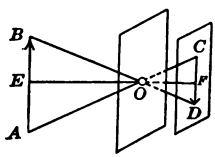
The point P is fixed. In making any adjustment, $\frac{PA}{PC}$ is made equal to $\frac{CB}{CS}$ and $ACBQ$ a parallelogram. Then P, Q, and S will always be in the same straight line. Prove that $\triangle PAQ \backsim \triangle PCS$ and that any movement of Q in a straight line is to the movement at S as $\frac{PA}{PC}$.

***5.** In the frame of the tower shown in the figure, determine the distances from A and B, C and D, etc., to make the holes in the braces so that they may be bolted at points a, b, c, etc. These distances should be accurate to tenths of an inch.

Ans. $Aa = 10$ ft. 5.3 in.



***6.** If the rays of light from any object AB pass through a small aperture O and fall upon a screen, an inverted image will be formed. A photograph may be taken in this manner if the sensitive plate takes the place of the screen. If the object is 6 ft. long, and is 10 ft. from the aperture, find the distance the screen must be placed from the aperture so that the image shall be 8 in. long.

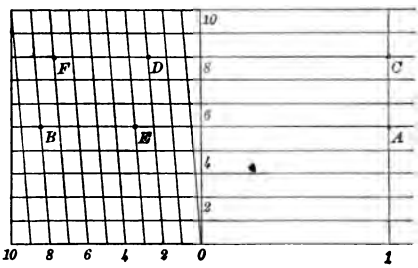


7. Construct a *diagonal scale* as shown in the figure. Use cardboard and make any width desired. By the use of this instrument, measurements to a hundredth of an inch can be made.

Show that $AB = 1.85$ in., and that $CD = 1.28$ in.

What is the length of AE? Of CF?

Show how to determine lines of the following lengths: 0.45 in., 0.76 in., 1.17 in., 1.47 in., 1.23 in.



In the United States there are in use three different kinds of V-shaped screw threads. *First*, the **Sharp V-Thread**, or **Common V-Thread**, in which the thread sides meet at a point both at the top and bottom, and make angles of 60° with each other as shown in Fig. 1.

Second, the **United States Standard Screw Thread** which is in general use. It is like the common V-thread, but has one-eighth of the height cut off from the top and the same amount filled in at the bottom. It is shown in Fig. 2.

Third, the **Whitworth Thread**, in which the angle between the sides of the threads is 55°, and the tops and bottoms are rounded, taking off one-sixth at the top and filling in the same amount at the bottom. It is shown in Fig. 3.

The **pitch** of a thread is the distance from the top of one thread to the top of the next.

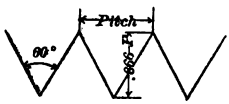


FIG. 1 *Sharp V- Thread*

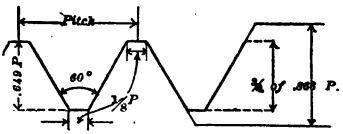


FIG. 2 *U.S. Standard Thread*

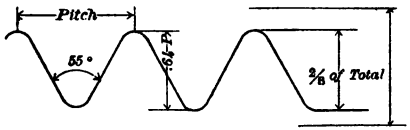


FIG. 3 *Whitworth Thread*

*8. Show that the depth of a common V-thread with a pitch of 1 in. is $\frac{1}{2}\sqrt{3}$ in., or 0.866 in.

*9. Show that the depth of any common V-thread is $\frac{0.866 \text{ in.}}{N}$, where N is the number of threads to an inch.

*10. Show that a U. S. standard thread with a pitch of 1 in. has a depth equal to 0.649 in., and that the depth of any U. S. standard thread is $\frac{0.649 \text{ in.}}{N}$, where N is the number of threads to an inch.

*11. Show that the depth of a Whitworth thread with a pitch of 1 in. is 0.64 in., and that the depth of any Whitworth thread is $\frac{0.64 \text{ in.}}{N}$, where N is the number of threads to an inch.

*12. Show that the root diameter, that is, the diameter of the screw at the bottom of the thread, for the different threads is given by the following formulas, where d is the root diameter, D the outer diameter, and N the number of threads per inch:

$$(1)\ d = D - \frac{1.732}{N}, \quad (2)\ d = D - \frac{1.3}{N}, \quad (3)\ d = D - \frac{1.28}{N}.$$

*13. The man in a machine shop often has to determine the taper per foot of a piece that is to be turned. By the taper per foot is meant the decrease in diameter per foot. In Fig. 1, the taper is $4\frac{1}{2}$ in. -3 in. $= 1\frac{1}{2}$ in. per foot. If l stands for the length of tapered part in feet, t for the taper in inches in this part, and T for the taper in inches per foot, show by similar triangles that the following proportion is true:

$$l : 1 = t : T.$$

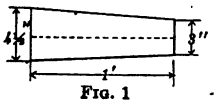


FIG. 1

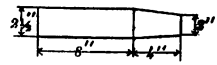


FIG. 2

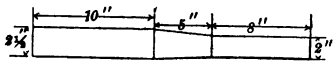
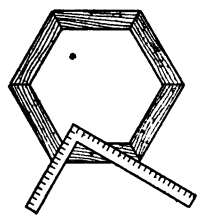


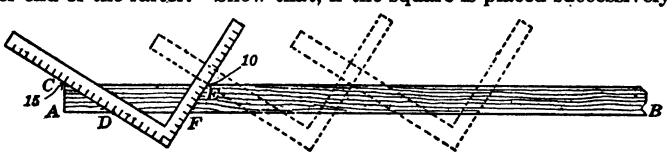
FIG. 3

*14. Find the taper per foot for the pieces shown in Fig. 2 and Fig. 3.

*15. Show how to mark the pieces for a hexagonal frame by the carpenter's square. If 6 in. is taken on the tongue of the square, how many inches should be taken on the blade?



*16. Show that if the carpenter's square is placed, as shown in the figure, upon the stick of timber AB from which a rafter with a run of 15 ft. and a rise of 10 ft. is to be made, the line CD gives the cut at the lower end of the rafter. Show that, if the square is placed successively 12



times as indicated, the length of the rafter will be determined, and that the line along the tongue in the last position will give the cut of the upper end of the rafter.

NOTE. The preceding Exercise gives a method for cutting *any* rafter.

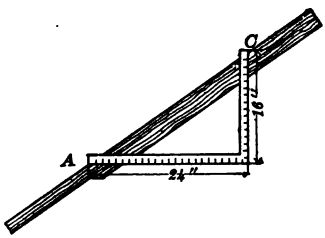


FIG. 1

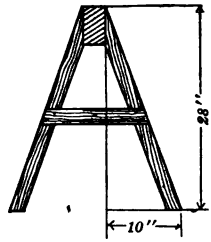


FIG. 2

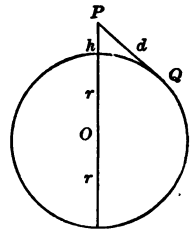


FIG. 3

***17.** Required to cut a rafter for a V-shaped roof in a building 18 ft. wide if the rise of the rafters is to be 6 ft. Each rafter is to be made of a piece of "two by four", and half its width is to project 18 in. beyond the plate.

SUGGESTION. The lower end of the rafter is cut as shown in Fig. 1.

***18.** Show how to mark the slants of the legs of the sawhorse shown in the figure. Compute the length of the legs. (Fig. 2.)

***19.** Fig. 3 represents the earth. Required to find the distance d one can see on the surface of the earth from a point P at a height h above the surface. Use r as the radius of the earth, and find that

$$d = \sqrt{2hr + h^2}.$$

***20.** What is the maximum distance at which an object at the surface of the water can be seen if the observer is 50 ft. above the water? Use 4000 miles as the radius of the earth.

***21.** The lamp of a lighthouse is 85 ft. above the surface of the water and the deck of a ship is 30 ft. above the surface of the water. Find the greatest distance at which the lamp can be seen from the deck of the ship.

Ans. 18 mi., approximately.

***22.** A cliff 2000 ft. high is on the seashore. How far away is the horizon?

REMARK. In Exercise 19, d, r, and h are in the same units. However, when h is small compared with r, a very simple approximate formula can be derived if h is taken in feet, r and d in miles, and $r = 3960$ miles. Then

$$d = \sqrt{2 \times 3960 \times \frac{h}{5280}} = \sqrt{\tfrac{3}{2}h}.$$

Or, solving for h, $h = \frac{2}{3}d^2$.

These formulas may be stated in the following rules:

The distance of the horizon in miles is equal to the square root of $\frac{3}{2}$ times the height of the point of observation in feet.

The height of the point of observation in feet is $\frac{2}{3}$ times the square of the distance of the horizon in miles.

***23.** How far above the surface of the earth must one be in order to see 50 miles? 100 miles?

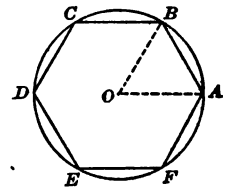
CHAPTER V

REGULAR POLYGONS. MEASUREMENT OF CIRCLES.

457. A regular polygon is a polygon that is both equilateral and equiangular.

458. Problem. *To inscribe a regular hexagon in a circle.*

Construction. Draw the radius OA. With A as a center and a radius equal to OA, describe an arc intersecting the circle at B. With B as a center and the same radius determine the point C, etc. Draw the chords AB, BC, etc. Then $ABCDEF$ is the required regular hexagon.



Proof. $\triangle AOB$ is equilateral. Why?

$\angle AOB = 60°$ and $\overset{\frown}{AB} = 60°$. Why?

Then the circle is divided into six equal arcs. Hence the chords are equal and the polygon $ABCDEF$ is equilateral.

The polygon is also equiangular, for each angle of the polygon equals in degrees one-half of four of the equal arcs into which the circle is divided.

$\therefore$ polygon $ABCDEF$ is a regular hexagon. Why?

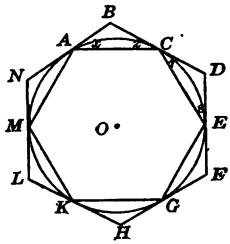
459. Theorem. ***Any equilateral polygon inscribed in a circle is a regular polygon.***

This is proved by the method that was used to prove that the equilateral hexagon is equiangular. § 458

460. Problem. *To inscribe a square in a circle.*

Draw two perpendicular diameters, complete the construction, and prove.

461. Theorem. ***If a circle is divided into any number of equal parts, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the successive points of division form a regular circumscribed polygon.***



Given the circle whose center is O divided into equal parts, the chords joining the successive points of division, and the tangents at these points.

To prove: (1) The inscribed polygon is regular.

(2) The circumscribed polygon is regular.

Suggestion. (1) Polygon $ACEGKM$ is equilateral by § 275, and therefore regular by § 459.

(2) $\angle x = \angle z = \angle r = \angle s$, etc. Why?

Show that $\triangle ABC = \triangle CDE = \triangle EFG$, etc., and that they are isosceles.

Then show that $BD = DF = FH$, etc.

$\therefore$ polygon $BDFHLN$ is regular. Why?

462. Theorem. *Tangents drawn to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

463. Theorem. *Chords drawn from the middle points of the arcs subtended by the sides of a regular inscribed polygon to the adjacent vertices of the polygon, form a regular inscribed polygon of double the number of sides.*

464. Theorem. *The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides, and less than the circumscribed circle.*

465. Theorem. *The area of a regular inscribed polygon is less than the area of a regular inscribed polygon of double the number of sides.*

466. Theorem. *The area of a regular polygon is less than the area within the circumscribed circle.*

467. Remark. From the considerations already made, it is evident that the general problem of constructing a regular polygon depends upon the division of a circle into as many equal parts as the polygon has sides. It is to be noted that this division of the circle cannot be made by methods of elementary geometry in all cases.

As early as the time of Euclid it was known that it is possible to construct regular polygons of 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, and $15 \cdot 2^n$ sides. It was long supposed that these were all that could be constructed.

Gauss, a famous German mathematician who lived from 1777 to 1855, however, proved that a circle can be divided into 17 equal parts by means of the instruments of geometry. He also solved the general problem as to what polygons can be inscribed in a circle by methods of elementary geometry. The results showed that the number of sides must be: (1) a *prime* number of the form $2^{2^n}+1$ (as 3, 5, 17, 257); or (2) a product of such prime numbers, not repeated, (as 15, 51, 85, 255); or (3) such prime numbers times 2^n; or (4) products of such prime numbers by 2^n.

EXERCISES

1. Show by § 460 and § 463 how to construct regular polygons of 8, 16, 32, 64 $\cdots$, sides.

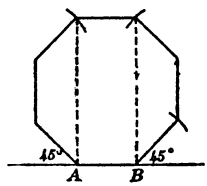
2. Show how to construct regular polygons of 12, 24, 48 $\cdots$, sides.

3. Show that an equilateral triangle may be formed by joining the alternate vertices of a regular hexagon.

4. Circumscribe a regular octagon about a circle.

5. On a given line as a side construct a regular octagon.

Let AB be the side. The figure suggests a construction.



6. If the diagonals joining the alternate vertices of a regular hexagon are drawn, they form a second regular hexagon whose area is one-third that of the original hexagon.

7. An equilateral polygon circumscribed about a circle is regular if the number of sides is odd.

8. An equiangular polygon circumscribed about a circle is regular.

9. Where are the vertices of all the regular polygons of the same number of sides which can be circumscribed about a given circle?

10. Where are the middle points of the sides of all regular polygons of the same number of sides which can be inscribed in a given circle?

11. Draw a regular hexagon, and draw all of its diagonals. Point out the equilateral triangles formed; the rhombuses; the right triangles; the rectangles; the isosceles triangles; the isosceles trapezoids. What other figures are there formed?

12. If r is the radius of a circle, prove the following:

(1) Inscribed equilateral triangle, side $= r\sqrt{3}$.

(2) Circumscribed equilateral triangle, side $= 2r\sqrt{3}$.

(3) Inscribed square, side $= r\sqrt{2}$.

(4) Circumscribed square, side $= 2r$.

(5) Inscribed regular hexagon, side $= r$.

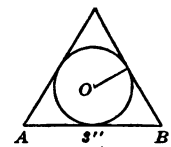
(6) Circumscribed regular hexagon, side $= \frac{2}{3}r\sqrt{3}$.

(7) Inscribed regular octagon, side $= r\sqrt{2-\sqrt{2}}$.

(8) Circumscribed regular octagon, side $= 2r(\sqrt{2}-1)$.

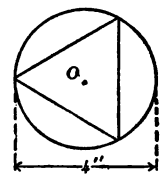
***13.** A square end 0.935 in. on a side must be made on a shaft. What is the diameter to which the shaft should be turned?

Ans. 1.322 in.



***14.** What is the diameter of the bearing that can be turned on a triangular shaft having a cross section that is 3 in. on a side? On a hexagonal shaft that is $1\frac{5}{8}$ in. on a side? *Ans.* 1.732 in.; 2.815 in.

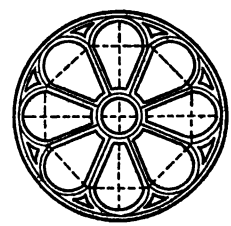
***15.** A round shaft has a diameter of 4 in. Find the length of the side of a triangular end that can be made on the shaft. Find the length of the side of a square end. Of a hexagonal end. Of an octagonal end.



16. Find the radius of a circle, if the difference of the sides of a regular inscribed and a regular circumscribed triangle is 12 in.

17. Find the radius of a circle, if the sum of the perimeters of regular inscribed and circumscribed hexagons is 100 in.

18. Construct the window design shown in the figure. Notice that it involves an octagon upon the sides of which semicircles are described.



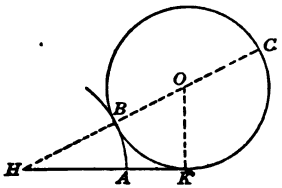
468. Extreme and Mean Ratio. When a line is divided into two segments so that one segment is the mean proportional between the whole line and the other segment, the line is said to be divided in **extreme and mean ratio.**

The line HK of the figure is divided in extreme and mean ratio when $\frac{HK}{HA}=\frac{HA}{AK}$.

469. Problem. *To divide a line in extreme and mean ratio.*

Given the line HK.

Required to divide HK in extreme and mean ratio.



Construction. Draw $OK \perp HK$ and equal to one-half HK.

Construct the circle whose center is O with a radius equal to OK.

Draw the secant HC passing through O and cutting the circle at B.

Lay off $HA = HB$. Then HK is divided at A in extreme and mean ratio.

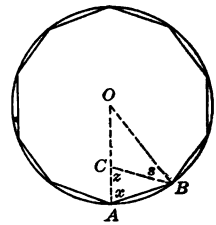
Proof.	$\frac{HC}{HK}=\frac{HK}{HB}$.	§ 450
Then	$\frac{HC-HK}{HK}=\frac{HK-HB}{HB}$.	§ 407
But	$HC-HK=HA$.	Why?
And	$HK-HB=AK$.	Why?
Whence	$\frac{HA}{HK}=\frac{AK}{HA}$	
And	$\frac{HK}{HA}=\frac{HA}{AK}$.	§ 404

$\therefore$ HK is divided at A in extreme and mean ratio.

470. Problem. *To construct a regular decagon in a given circle.*

Given the circle with center O.

Required to inscribe a regular decagon in the given circle.



Construction. Draw the radius OA.

Divide OA in extreme and mean ratio so that $\frac{OA}{OC}=\frac{OC}{CA}$. § 469

With A as center and a radius equal to OC, describe an arc intersecting the circle at B. Draw the chord AB, which will be a side of the required decagon.

Proof. Draw OB and CB.

Then $\triangle ABC \backsim \triangle AOB$. § 431

For $\angle x$ is common to the two triangles and, since $AB=OC$,

$$\frac{OA}{AB}=\frac{AB}{CA}.$$

Since $\triangle AOB$ is isosceles, $\triangle ABC$, also, is isosceles and $\angle z=\angle x$.

Then $AB=BC=OC$, and $\triangle BCO$ is isosceles.

Hence $\angle s=\angle O$.

Now $\angle z=\angle s+\angle O=2\angle O$. Why?

And $\angle x=\angle ABO=2\angle O$. Why?

Then $\angle x+\angle ABO+\angle O=5\angle O$. Why?

But $\angle x+\angle ABO+\angle O=180°$, Why?

Whence $5\angle O=180°$, and $\angle O=36°$.

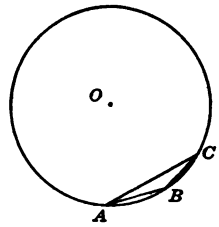
Therefore $\overset{\frown}{AB}=36°$, or $\frac{1}{10}$ of the circle.

If then AB is applied successively to the circle as a chord, a regular decagon will be formed.

471. Problem. *To inscribe a regular pentagon in a circle.*

Join the alternate vertices of a regular decagon.

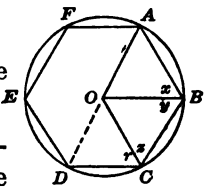
472. Problem. *To inscribe in a given circle a regular pentadecagon, or polygon of fifteen sides.*



Discussion. O is the center of the given circle in which the pentadecagon is to be inscribed. Draw AC the side of a regular inscribed hexagon and AB the side of a regular inscribed decagon. $\overset{\frown}{AB}$ is $\frac{1}{10}$ of the circle, and $\overset{\frown}{AC}$ is $\frac{1}{6}$ of the circle.

Then $\overset{\frown}{BC}$ is $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$, of the circle. Therefore, if BC is applied successively to the circle as a chord, a regular pentadecagon will be inscribed.

473. Problem. *To circumscribe a circle about any regular polygon.*



Given $ABCDEF$, a regular polygon.

Required to circumscribe a circle about the polygon.

Outline of construction and proof. Construct the arc of a circle passing through the three vertices A, B, and C. From O, the center of this arc, draw the radii OA, OB, and OC. Also draw OD.

Now	$OB=OC$, and $AB=CD$.	Why?
Also	$\angle ABC=\angle BCD$ and $\angle y=\angle z$.	Why?
Then	$\angle x=\angle r$.	Why?
	$\triangle OBA=\triangle OCD$.	Why?
And	$OD=OA$.	Why?

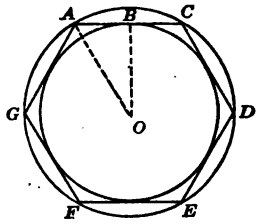
Hence arc ABC if produced will pass through the vertex D.

In like manner it can be shown that the arc, if produced, will pass through all the vertices of the polygon and therefore be circumscribed about the polygon.

474. Problem. *To inscribe a circle in any regular polygon.*

Given *ACDEFG*, a regular polygon.

Required to inscribe a circle in the polygon.



Outline of construction and proof. First circumscribe a circle about the polygon. § 473

Since the sides of the polygon are chords of the circumscribed circle they are all equidistant from the center *O*. Why?

Construct $OB \perp AC$. With *O* as a center and *OB* as a radius, construct the circle. This circle will be inscribed in the polygon. Why?

475. Definitions. The **center of a regular polygon** is the common center of its inscribed and circumscribed circles.

The **radius of a regular polygon** is the radius of the circumscribed circle.

Thus in the figure of § 474, *OA* is the radius of the polygon.

The **apothem of a regular polygon** is the radius of the inscribed circle.

Thus *OB* is the apothem of the polygon.

EXERCISES

1. Show how to construct regular polygons of 20, 40, 80, $\cdots$ $5 \cdot 2^n$ sides.

2. Show how to construct regular polygons of 30, 60, $\cdots$ $15 \cdot 2^n$ sides.

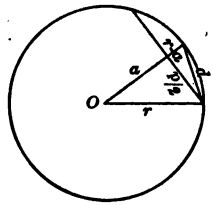
3. Construct angles of 22° 30′, 11° 15′, 36°, 72°, 18°, 9°, 30°, 15°, 7° 30′.

4. Is every equiangular polygon inscribed in a circle regular? Prove.

***5.** Derive a formula for computing the side of a regular inscribed decagon if the radius r of the circle is given. *Ans.* $\frac{1}{2}r(\sqrt{5}-1)$.

*6. The apothem of a regular inscribed pentagon is equal to half the sum of the radius of the circle and the side of the regular decagon inscribed in the same circle.

Let r = radius, d = side of decagon, p = side of pentagon, and a = apothem.



(1) $a^2 = r^2 - (\frac{1}{2}p)^2$.

(2) $(r-a)^2 = d^2 - (\frac{1}{2}p)^2$.

Subtracting (2) from (1),

(3) $2ar - r^2 = r^2 - d^2$.

(4) But $\frac{r}{d} = \frac{d}{r-d}$. Why?

(5) Then $rd = r^2 - d^2$.

(6) Substituting in (3), $2ar - r^2 = rd$. Therefore $a = \frac{r+d}{2}$.

*7. The square of the side of a regular inscribed pentagon equals the sum of the squares of the sides of the regular inscribed decagon and the radius of the circle.

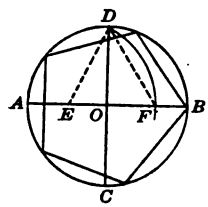
Use the notation of the last exercise.

$$r^2 = \left(\frac{p}{2}\right)^2 + \left(\frac{r+d}{2}\right)^2. \quad d^2 = \left(\frac{p}{2}\right)^2 + \left(\frac{r-d}{2}\right)^2.$$

Adding, $$r^2 + d^2 = \frac{p^2}{2} + \frac{r^2+d^2}{2}.$$

$$\therefore p^2 = r^2 + d^2.$$

*8. A convenient way to lay out a regular pentagon in a circle is to draw two perpendicular diameters AB and CD; bisect AO at E; and, with E as center and ED as radius, draw the arc DF. The length DF is equal to the side of the inscribed pentagon, which can then be laid out.



To prove that DF is the side of the inscribed pentagon, proceed as follows:

$ED = \sqrt{r^2 + (\frac{1}{2}r)^2} = \frac{1}{2}r\sqrt{5}$.

Then $EF = \frac{1}{2}r\sqrt{5}$, and $OF = EF - EO = \frac{1}{2}r(\sqrt{5}-1)$.

Hence $OF = d$, the side of the inscribed decagon.

Now $DF = \sqrt{\overline{OD}^2 + \overline{OF}^2} = \sqrt{r^2 + d^2}$.

$\therefore DF = p$, the side of inscribed pentagon.

NOTE. This Exercise also suggests another construction for the side of a regular inscribed decagon. The construction is due to Ptolemy of Alexandria, one of the greatest Greek astronomers, who died in 165 A. D.

9. Construct the star design which is based upon a regular pentagon.

10. Prove that the diagonals of a regular pentagon, by their points of intersection, determine another regular pentagon.

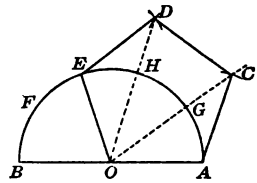
11. Prove that any two diagonals not drawn from the same vertex in a regular pentagon, divide each other in extreme and mean ratio.

***12.** Derive a formula for computing the side of a regular inscribed pentagon when the radius r of the circle is known. *Ans.* $\frac{1}{2}r\sqrt{10-2\sqrt{5}}$.

13. Construct upon a given line as one side, each of the following regular polygons:

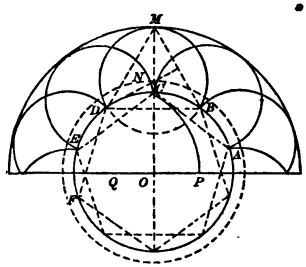
(1) triangle, (2) quadrilateral, (3) pentagon,
(4) hexagon, (5) octagon, (6) decagon,
(7) dodecagon, (8) pentadecagon.

Construction of the Pentagon.



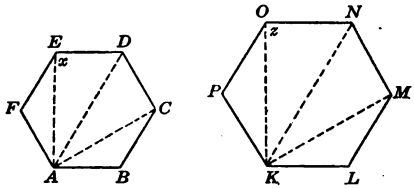
Let OA be the length of a side of the pentagon. With O as center and OA as radius describe a semicircle, and extend AO to B. With the protractor, or otherwise, divide the semicircle into five equal parts at the points G, H, E, and F. Draw lines from O through the points E, H, and G. With OA as radius, the points C and D can readily be found. $OEDCA$ is the required pentagon. Give proof.

NOTE. The angle between two adjacent sides of any regular polygon of n sides is $\frac{(n-2)180^\circ}{n}$. The method outlined above for the pentagon can then be used to construct a regular polygon of any number of sides.



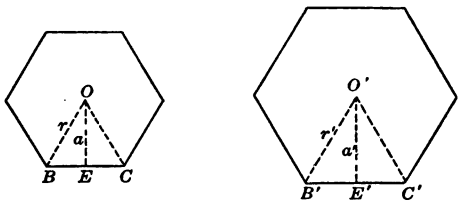
***14.** Construct the window design shown in the figure. A, C, E, $\cdots$ are the vertices of a regular inscribed pentagon; as also are B, D, F, $\cdots$. The small circles circumscribe equilateral triangles constructed on the sides of the pentagons.

476. Theorem. *Two regular polygons of the same number of sides are similar.*



Draw diagonals from one vertex in each and prove that the polygons are divided into the same number of triangles, similar each to each and similarly placed.

477. Theorem. *The perimeters of two similar regular polygons are to each other as their radii, and also as their apothems.*



Given two similar regular polygons, the apothems a and a', the radii r and r', and the perimeters p and p'.

To prove $$\frac{p}{p'}=\frac{r}{r'}=\frac{a}{a'}.$$

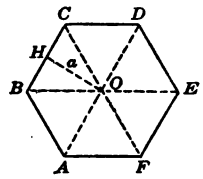
Proof. Show that $\triangle OBC$ and $\triangle O'B'C'$ are similar. § 424

Then $$\frac{BC}{B'C'}=\frac{r}{r'}=\frac{a}{a'}.$$ Why?

But $$\frac{p}{p'}=\frac{BC}{B'C'}.$$ § 445

$$\therefore \frac{p}{p'}=\frac{r}{r'}=\frac{a}{a'}.$$ Why?

478. Theorem. *The area of a regular polygon is equal to half the product of the perimeter by the apothem.*



Given $ABCDEF$ a regular polygon, of which A is the area, a the apothem, and p the perimeter.

To prove that $A = \frac{1}{2}ap$.

Outline of proof. Draw OA, OB, etc., the radii of the polygon. Show that these divide the polygon into as many equal triangles as the polygon has sides. Also that the area of $\triangle OBC = \frac{1}{2}a \times BC$. Then, since the altitude of each triangle is the apothem of the polygon, the sum of the bases of the triangles is the perimeter of the polygon, and the sum of the areas of the triangles is the area of the polygon,

$$\therefore A = \tfrac{1}{2}ap.$$

479. Theorem. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii, and also as the squares of their apothems.*

EXERCISES

1. Prove that the area of a regular inscribed hexagon is the mean proportional between the areas of inscribed and circumscribed equilateral triangles.

2. Each of three figures, an equilateral triangle, a square, and a hexagon, has an area of 1000 sq. in. Which figure has the least perimeter? The greatest? What is the perimeter of each?

3. What is the ratio of the apothems of two regular octagons whose areas are in the ratio of 1 : 2? What is the ratio of their sides?

4. Construct two concentric octagons whose areas are in the ratio of 1 : 3.

5. If r is the radius of a circle prove the following formulas:

(1) Inscribed equilateral triangle, area $= \frac{3}{4}r^2\sqrt{3}$.
(2) Circumscribed equilateral triangle, area $= 3r^2\sqrt{3}$.
(3) Inscribed square, area $= 2r^2$.
(4) Circumscribed square, area $= 4r^2$.
(5) Inscribed regular hexagon, area $= \frac{3}{2}r^2\sqrt{3}$.
(6) Circumscribed regular hexagon, area $= 2r^2\sqrt{3}$.

MEASUREMENT OF THE CIRCLE

480. Length of Circle. By the **length of a circle** (circumference) is meant the numerical measure of the circle, that is, the number of linear units in it.

The general idea of the measurement of a straight line cannot be used here, for it is impossible to apply a unit straight line to a circle and thus find its numerical measure. Practically, however, the length of a circle, such as a wheel, may be determined by wrapping a tape line about it, that is, by considering the circle as straightened out and then measured. In this manner, the approximate length of a circle can be determined; but the method is not exact, nor does it depend upon geometric relations. It remains, then, to devise a method for measuring a circle which is both accurate and geometric.

481. A **constant** is a number whose value does not change.

The number of inches in a foot is a constant.

482. A **variable** is a number whose value is continually changing.

The distance of a moving train from a station it has recently left is a variable. The length of a chord of a circle, as the chord rotates about one of its points, is a variable.

483. Limit of a Variable. When a variable changes in such a manner that its successive values approach a constant so nearly that the difference between the constant and the variable becomes and remains less than any assigned positive number, however small, the constant is said to be the **limit of the variable.**

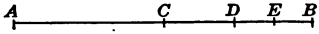
The variable is said to *approach a constant as a limit.* If the variable is represented by x and the constant by a, then the statement "x approaches a as a limit" is written thus: $x \longrightarrow a$.

The definition implies that not all variables have limits. It is also true that while some variables become equal to their limits, other variables never do.

484. Examples of Variables and Limits. (1) The variable which takes the successive values 1.3, 1.33, 1.333, $\cdots$ has as a limit $1\frac{1}{3}$. That is, the more figures there are taken, the more nearly the number approaches $1\frac{1}{3}$.

(2) The $\sqrt{2}$ is the limit of the successive values 1.4, 1.41, 1.414, 1.4142, $\cdots$. The diagonal of a unit square is the limit of the *line lengths* represented by this series of numbers.

(3) If a point starts at the end A of the line AB and during the first second moves half the length of the line to C; during the next second, half the remaining distance to D; continuing in this way to move half the remaining distance during each successive second, then the distance that the point is from A is a variable of which AB is the limit.



485. Theorems on Limits. The following are used in propositions which follow, and are given here without proof:

(1) *If two variables that approach limits are equal for all their successive values, their limits are equal.*

(2) *If x is a variable having the limit a, and c is a constant not zero, then $cx \to ca$ and $\frac{x}{c} \to \frac{a}{c}$ as $x \to a$.*

EXERCISES

1. Does the height of a tree have a limit?

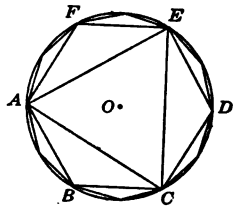
2. If the point moves as stated in § 484, (3), how many seconds will it take for the point to approach within $\frac{1}{100}$ in. of B, AB being 2 in.? To approach within 0.001 in. of B?

3. Does the length of a chord revolving about a point within a circle have a limit? Does it reach its limit?

4. Give several illustrations of variables and limits.

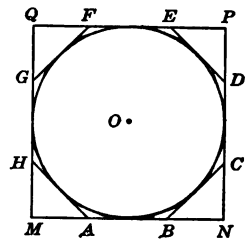
5. What is the limit of the series of numbers $2\frac{1}{2}$, $2\frac{1}{4}$, $2\frac{1}{8}$, $2\frac{1}{16}$, $\cdots$?

486. In the circle whose center is O, a regular polygon is inscribed and the number of sides is doubled indefinitely according to § 463. The circle is a constant and the perimeter of the polygon is a variable. The perimeter of each polygon is greater than the perimeter of the polygon of half the number of sides. § 464



At all times, each arc of the circle is greater than the subtending chord (why?), and the circle is greater than the perimeter of each polygon, § 464. By doubling the number of the sides of the polygon indefinitely, the perimeter steadily increases, and can be made to differ from the circle by less than any assigned value, that is, *it can be made just as nearly equal to the circle as desired.*

Similarly, by circumscribing a regular polygon about a circle and doubling the number of its sides indefinitely, the perimeter steadily decreases, and can be made to differ from the circle by less than any assigned value, that is, *it can be made just as nearly equal to the circle as desired.*



487. It is evident that the above method can be as readily applied to the area of a circle as to its length.

By the **area of a circle** is meant the area within the circle.

It is thus seen that the area of an inscribed, or circumscribed, regular polygon, as the number of sides is doubled indefinitely, *can be made just as nearly equal to the area of the circle as desired.*

The above discussions establish the following statements:

488. *The length of a circle is the common limit of the perimeters of regular inscribed and regular circumscribed polygons as the number of sides is indefinitely doubled.*

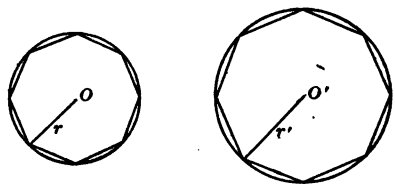
489. *The area of a circle is the common limit of the areas of regular inscribed and regular circumscribed polygons as the number of sides is indefinitely doubled.*

490. Theorem. *The apothem of a regular inscribed polygon approaches as a limit the radius of the circle, as the number of sides is indefinitely doubled. Also, the radius of a regular circumscribed polygon approaches as a limit the radius of the circle, as the number of sides is indefinitely doubled.*

491. The length of a circle is called the **circumference.**

492. Theorem. ***Two circumferences have the same ratio as their radii, or as their diameters.***

Given the two circles O and O'. Let c and c' denote their circumferences, r and r' their radii, and d and d' their diameters.



To prove $\dfrac{c}{c'}=\dfrac{r}{r'}=\dfrac{d}{d'}$.

Proof. Let p and p' denote the perimeters of the similar regular polygons inscribed in the circles.

$$\frac{p}{p'}=\frac{r}{r'}. \qquad \text{§ 477}$$

By doubling indefinitely the number of the sides of the polygons, $p \to c$ and $p' \to c'$.

Then $\dfrac{p}{r} \to \dfrac{c}{r}$ and $\dfrac{p'}{r'} \to \dfrac{c'}{r'}$. § 485 (2)

But $\dfrac{p}{r}=\dfrac{p'}{r'}$, being variables that are always equal. § 477

Then $\dfrac{c}{r}=\dfrac{c'}{r'}$. § 485 (1)

And $\dfrac{c}{c'}=\dfrac{r}{r'}$. § 405

But $\dfrac{r}{r'}=\dfrac{d}{d'}$. Why?

$\therefore \dfrac{c}{c'}=\dfrac{r}{r'}=\dfrac{d}{d'}$. Why?

493. Theorem. *The ratio of a circumference to its diameter is constant.*

It follows directly from § 492 that the ratio of the length of a circle to its diameter is the same for any two circles, that is, the ratio is *constant.*

494. The Number π. The constant ratio of the circumference of a circle to its diameter is represented by the Greek letter π (pronounced pī), that is, $\frac{c}{d}=\pi$.

495. Theorem. ***The circumference of any circle is equal to πd, or to $2\pi r$.***

Proof. $\frac{c}{d}=\pi$. § 494

$\therefore c=\pi d.$

But $d=2r.$

$\therefore c=2\pi r.$

496. Theorem. *If an arc s is intercepted by a central angle of u degrees, the length of the arc is $\frac{u}{360}\cdot\pi d$.*

Proof. $\frac{\text{arc } s}{c}=\frac{u}{360}.$

$\therefore \text{arc } s=\frac{u}{360}\cdot c=\frac{u}{360}\cdot\pi d.$

EXERCISES

1. Construct a circle equal in length to two given circles of radii r_1 and r_2 respectively. Of diameters d_1 and d_2 respectively.

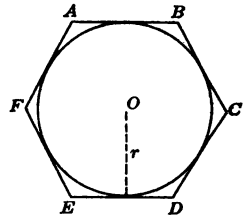
Let r = radius of the required circle.

Then $2\pi r=2\pi r_1+2\pi r_2.$ $\therefore r=r_1+r_2.$

2. Construct a circle equal in length to any number of given circles.

3. Construct a circle equal in length to the difference between two given circles.

497. Theorem. ***The area of a circle is equal to one-half the product of the circumference by the radius.***



Given a circle with center O. Let c denote the circumference, r the radius, and A the area.

To prove $A=\frac{1}{2}rc$.

Proof. Circumscribe any regular polygon as $ABCDEF$ about the circle, and let A' denote its area and p its perimeter. When the number of sides of the polygon is doubled indefinitely, A' and p become variables.

Also	$p \to c$ and $A' \to A$.	§§ 488, 489
Then	$\frac{1}{2}rp \to \frac{1}{2}rc$.	§ 485 (2)
But it is always true that	$A'=\frac{1}{2}rp$.	§ 478
	$\therefore A=\frac{1}{2}rc$.	§ 485 (1)

498. Theorem. ***The area of any circle is equal to πr^2.***

Proof.	$A=\frac{1}{2}rc$.	§ 497
But	$c=2\pi r$.	§ 495
Then	$A=\frac{1}{2}r(2\pi r)$.	
	$\therefore A=\pi r^2$.	

499. Theorem. *The area of any circle is equal to $\frac{1}{4}\pi d^2$.*

$r=\frac{1}{2}d$ and $c=\pi d$. § 495

$\therefore A=\frac{1}{2}(\frac{1}{2}d)(\pi d)=\frac{1}{4}\pi d^2$.

500. Theorem. *The areas of two circles are in the same ratio as the squares of their radii or as the squares of their diameters.*

Proof. Let A and A', r and r', and d and d' denote the areas, radii, and diameters respectively of the two circles.

Then $\frac{A}{A'}=\frac{\pi r^2}{\pi r'^2}$. Why?

$\therefore \frac{A}{A'}=\frac{r^2}{r'^2}=\frac{d^2}{d'^2}$. Why?

EXERCISES

1. Using c for circumference, d for diameter, r for radius, and A for area of a circle, show that the following formulas are true:

(1) $d = c \div \pi$. (2) $r = c \div 2\pi$.

(3) $d = \sqrt{A \div \frac{1}{4}\pi}$. (4) $r = \sqrt{A \div \pi}$.

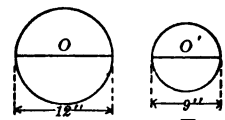
2. The area of a circle having a radius of 23 ft. is 1661.9 sq. ft. Find the area of a circle having a radius of 69 ft.

3. Construct a circle having an area equal to the area of two given circles of radii r_1 and r_2 respectively. Of diameters d_1 and d_2 respectively.

SUGGESTION. Let r = radius of required circle. Then r may be taken as the hypotenuse of a right triangle of which r_1 and r_2 are the sides.

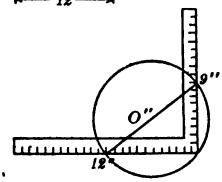
4. Construct a circle equal in area to the areas of any number of given circles. See Ex. 2, p. 175.

5. Construct a circle equal in area to the difference in the areas of two given circles.

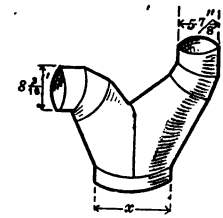


***6.** Show how the carpenter's square can be used to find the diameter of a circular pipe having the same area of cross section as the sum of the areas of two or more given circular pipes.

SUGGESTION. The above may be considered in this way: To find the diameter of a pipe to carry the hot air from a 9 in. pipe and a 12 in. pipe, place a ruler connecting the point 9 in. off on one arm of a square and 12 in. off on the other, and measure the distance between the points. This distance is the required diameter. Why?



***7.** By means of the carpenter's square, determine the diameter x of a pipe having the same area in cross section as two pipes $5\frac{7}{8}$ and $8\frac{3}{16}$ inches in diameter respectively.



***8.** Establish the following formulas for finding the area of a ring:

(1) $A = \pi R^2 - \pi r^2$.

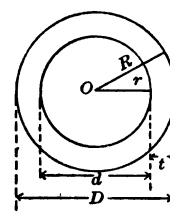
(2) $A = \pi(R+r)(R-r)$.

(3) $A = \frac{1}{4}\pi(D^2 - d^2)$.

(4) $A = \frac{1}{4}\pi(D+d)(D-d)$.

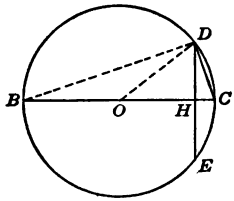
(5) $A = \frac{1}{2}\pi t(D+d)$.

(6) $A = \pi t(D-t)$.



COMPUTATION OF π

501. Problem. *Having given the length of the side and the radius of a regular inscribed polygon, find the length of the side of a regular inscribed polygon having double the number of sides.*



Given DE a side of a regular polygon inscribed in the circle whose radius is OC.

Required to find DC, a side of a regular inscribed polygon having double the number of sides.

Denote the radius OC by r, and the side DE by a. Draw the diameter $BC \perp DE$, and draw BD and OD.

$DH = \frac{1}{2}a.$ Why?

And $\overline{OH}^2 = \overline{OD}^2 - \overline{DH}^2$ § 377

$= r^2 - \frac{1}{4}a^2.$

Then $OH = \sqrt{r^2 - \frac{1}{4}a^2}.$

Also $HC = r - OH$

$= r - \sqrt{r^2 - \frac{1}{4}a^2}.$

And $\overline{DC}^2 = BC \times HC$ § 435

$= 2r\left(r - \sqrt{r^2 - \frac{1}{4}a^2}\right)$

$= 2r^2 - r\sqrt{4r^2 - a^2}.$

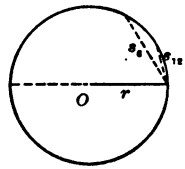
$\therefore DC = \sqrt{2r^2 - r\sqrt{4r^2 - a^2}}.$

502. Theorem. *When* $r = 1$, $DC = \sqrt{2 - \sqrt{4 - a^2}}.$

503. Problem. *To find the approximate numerical value of* π.

Given the circle O, with circumference c and radius r.

Required to find the approximate numerical value of π.



$$c = 2\pi r. \qquad \text{§ 495}$$

Then $$\pi = \frac{c}{2r}.$$

When $r = 1$, $\pi = \frac{1}{2}c$, and $c = 2\pi$.

Let s_6 be one side of a regular inscribed polygon of 6 sides, s_{12} of 12 sides, s_{24} of 24 sides, and so on.

If $r = 1$, $s_6 = 1$. § 458

By doubling the number of sides indefinitely, the difference between the length of the circle and the perimeter of the polygon may be made as small as desired. § 486

By § 502 we have:

	Computation	Length of Side	Perimeter
s_6	$= 1$	$= 1.00000000$	6.00000000
s_{12}	$= \sqrt{2-\sqrt{4-1}}$	$= 0.51763809$	6.21165708
s_{24}	$= \sqrt{2-\sqrt{4-(0.51763809)^2}}$	$= 0.26105238$	6.26525722
s_{48}	$= \sqrt{2-\sqrt{4-(0.26105238)^2}}$	$= 0.13080626$	6.27870041
s_{96}	$= \sqrt{2-\sqrt{4-(0.13080626)^2}}$	$= 0.06543817$	6.28206396
s_{192}	$= \sqrt{2-\sqrt{4-(0.06543817)^2}}$	$= 0.03272346$	6.28290510
s_{384}	$= \sqrt{2-\sqrt{4-(0.03272346)^2}}$	$= 0.01636228$	6.28311544
s_{768}	$= \sqrt{2-\sqrt{4-(0.01636228)^2}}$	$= 0.00818121$	6.28316941

Here the computation has been continued until further computation will not change the first four decimal places.

Then $c = 6.283169$, approximately.

$\therefore \pi = 3.1416$, approximately.

504. Remark. It should be noted that the value of π could be as readily approximated by starting with some other regular inscribed polygon, for instance, the square.

Further, it should be noted that π could be approximated by using a regular circumscribed polygon, for, by § 486, the perimeter of a regular circumscribed polygon approaches the circumference of the circle as the number of its sides is continually doubled.

505. The Quadrature of the Circle. One of the noted problems of antiquity was that of the *quadrature of the circle*, or *squaring the circle*, that is, the problem which requires the finding of a square which shall be equal in area to that of a circle with a given radius.

Since the area of a circle equals half its circumference multiplied by its radius, an equivalent square could be constructed geometrically by § 390 if the circumference could be found when the radius is known, or *vice versa*. To accomplish this it would be necessary to find the *exact* ratio of the circumference to the diameter of a circle. But this is impossible for it has been proved in higher mathematics that the ratio of the circumference to the diameter of a circle is transcendental.

506. Historical Note. The people of Babylon and the early Hebrews (I Kings vii, 23) used 3 as the value of π. Perhaps the earliest known attempt to find accurately the area of a circle was made by Ahmes of Egypt about 1700 B.C. His method gave a value for π which, in our notation, equals 3.1604.

Archimedes (287-212 B.C.), by a method similar to that of § 503, found that π must be between $3\frac{1}{7}$ and $3\frac{10}{71}$.

Metius of Holland (1571-1635), found $\pi = \frac{355}{113} = 3.1415929$.

Vieta of France (1540-1603), carried the approximation to nine places.

Ludolph van Ceulen of Holland (1540-1610), computed π to 35 places.

Others who computed π were Sharp in 1705 to 72 places, Machin in 1706 to 100 places, DeLagny in 1719 to 127 places, Rutherford in 1841 to 208 places, and in 1853 to 440 places, and Shanks in 1873 to 707 places.

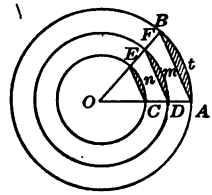
In 1761, Lambert proved that π is not rational. In 1882, Lindemann proved that π is not algebraic, that is, cannot be a root of an algebraic equation. This amounts to proving that it is impossible to construct geometrically a line equal to the circumference of a given circle, or a square equal in area to that of a given circle.

The following is the value of π to 42 decimal places:

$\pi = 3.14159265358979323846264338327950288419716 9 \cdots$.

AREAS OF SECTORS AND SEGMENTS

507. Definitions. A **sector** is the area bounded by two radii and their intercepted arc.

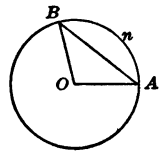


A **segment** of a circle is the area bounded by an arc and its chord.

Similar arcs, similar sectors, and **similar segments** are those which have equal central angles in circles.

508. Theorem. *The area of a sector is equal to half the product of its radius by its arc.* That is, $S=\frac{1}{2}rs$, if $S=$area, $r=$radius, and $s=$arc.

509. Area of Segment. In many practical problems it is necessary to find the area of a segment. In general, this cannot be done by geometry alone.

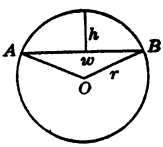


It is evident that the area of a segment, when the angle is acute or obtuse, is equal to the area of the sector minus the area of the triangle formed by the radii and the chord.

Area of segment ABn = area of sector AOB − area of triangle AOB.

Whether or not the area of a segment can be found by geometry depends upon what parts are given. Numerous approximate formulas can be given for the area of a segment. The following are two of them:

Using G for area of segment, w for width AB, and h for height, (1) $G=\frac{2}{3}hw$. (2) $G=\frac{2}{3}hw+\frac{h^3}{2w}$.

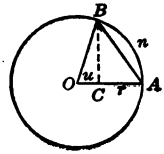


Formula (1) should be used only when $h < \frac{1}{10}r$. Formula (2) is always more nearly accurate than (1).

The following formula which involves trigonometry gives the area of a segment accurately:

Using G for area of segment, S for area of sector, u for angle of sector, and r for radius of circle, then

$$G=S-\tfrac{1}{2}r^2 \sin u.$$



Proof. (When u is acute.)

$$G=S-\triangle AOB.$$

But $\triangle AOB=\frac{1}{2}OA\cdot CB.$

Also $OA=r$, and $CB=OB\sin u=r\sin u$,

$$\therefore G=S-\tfrac{1}{2}r^2 \sin u.$$

EXERCISES

1. Find the circumferences and areas of circles with radii as follows: (1) 5 in. (2) 25 ft. (3) 7.21 in. (4) 4 ft. 8 in.

2. Find the diameters and circumferences of circles with areas as follows: (1) 16π sq. in. (2) 100 sq. in. (3) 785.4 sq. in.

3. Find the diameters and areas of circles with circumferences as follows: (1) 32.9868 ft. (2) 976 in. (3) 100 meters.

4. One of the large trees of California measures 90 ft. in circumference; find its diameter.

5. If a man can walk $3\frac{1}{2}$ miles an hour, how long will it take him to cross a circular island having a circumference of 52 miles?

6. A fly-wheel 30 ft. in diameter is running at 35 revolutions per minute. Find the speed in feet per minute of the rim of the fly-wheel.

7. A shaft is 100 ft. deep. How many coils of rope will be required to reach the bottom, the diameter of the roller on which the rope is wrapped being 11 in., and the diameter of the rope 1 inch?

8. The diameter of the high pressure cylinder of a steam engine is 30 in. and that of the low pressure cylinder is 64 in. Find the ratio of the areas of the two cylinders.

9. How is the area of a circle affected by multiplying the radius by 3? By 5? By n?

10. If the diameter of the drive wheels of a locomotive is 66 in., find the number of revolutions per minute they make when the engine is going 40 miles per hour.

11. The minute hand of a tower clock is 6 ft. long. How far will the extremity move in 36 minutes? *Ans.* 22 ft. 7.4 in.

12. How many revolutions per minute would a wheel of diameter 56 in. have to make in order to travel at a rate of 30 miles per hour?

13. Find the approximate length of wire in a coil having 30 turns if the coil is 3 ft. in diameter.

14. Find the area of a sector having a radius of 17 in. and an arc of 21 in.

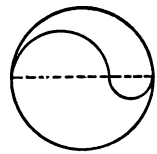
15. Use the approximate formulas and find the area of a segment 14 in. wide and 3 in. high.

16. Find the area of a segment whose central angle is 60° in a circle having a radius of 16 in.

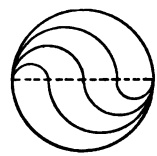
***17.** Find the area of a segment whose central angle is 70° in a circle having a radius of 10 ft.

GENERAL EXERCISES

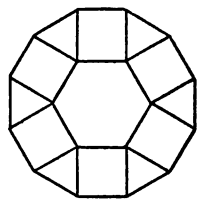
THEOREMS



1. If the diameter of a circle is divided into two parts, and upon these parts semicircumferences are described on opposite sides of the diameter, the semicircumferences will divide the circle into two parts which have the same ratio as the two parts of the diameter.



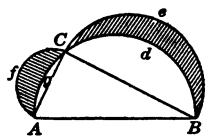
2. In the figure, the diameter is divided into four equal parts. The area of the circle is divided as shown by semicircumferences drawn on opposite sides of the diameter. Show that the area of the circle is divided into four equal parts.



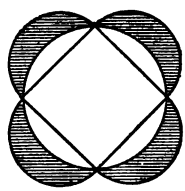
3. If squares are constructed outwardly upon the sides of a regular hexagon, their exterior vertices will be the vertices of a regular dodecagon.

4. The area of a regular dodecagon equals three times the square of the radius.

5. A rule sometimes given for finding the area of a ring between two concentric circles is as follows: The area of the ring is equal to the area of a circle whose diameter is that chord of the outer circle which is tangent to the inner circle. Is the rule correct? Prove it.



6. The crescents $AfCg$ and $CeBd$ are formed by constructing semicircles on the three sides of a right triangle. Show that the sum of the areas of the crescents equals the area of the triangle.

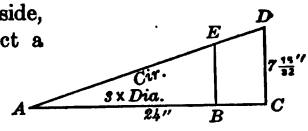


7. If, on the sides of an inscribed square as diameters, semicircles are described, the area of the four crescents lying outside the circle equals the area of the inscribed square.

NOTE. A great Greek geometrician, Hippocrates (430 B.C.) tried to "square the circle" by means of this theorem.

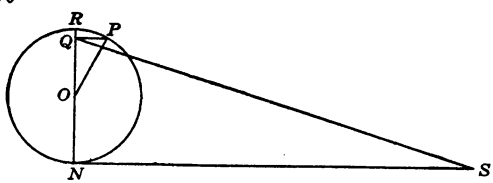
CONSTRUCTIONS

1. The following rule is given for finding the circumference of a circle: Take a right triangle of sides 24 in. and $7\frac{5}{12}$ in. respectively, lay off 3 times the diameter of the circle on the longer side, starting from the acute angle, and erect a perpendicular to meet the hypotenuse. The part of the hypotenuse from the acute angle to this intersection is the length of the circumference. What is the per cent of error in this construction?

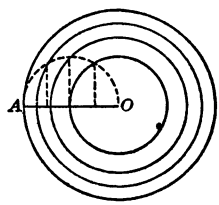


2. Justify the following approximate construction for determining the circumference of a circle when the diameter is given, and show that the error is about 1 in 13,000,000. Draw two perpendicular lines AB and AC. On AB lay off $AD = 7\frac{1}{16}$ in. and on AC lay off $AE = 22\frac{3}{16}$ in. Draw DE. Also on AB lay off AF equal to the given diameter and draw a line through F parallel to DE intersecting AC in G. Then the length AG equals the circumference. Show that this construction uses the ratio of diameter to circumference 113 : 355.

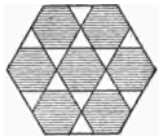
***3.** The following is another method for constructing the length of a circumference: Draw NS perpendicular to diameter NR and equal to three times the diameter. Draw OP making angle $POR = 30°$. Draw QP perpendicular to NR. Then QS is the circumference. The error is about $\frac{1}{20000}$. Show this.



4. To divide a given circle into any number of equal or proportional parts by concentric circles, the following construction may be made: Construct the given circle and divide the radius into the desired number of equal, or proportional parts. Construct a semicircle on the radius as diameter and erect perpendiculars at the points of division to intersect the circumference of the semicircle. Construct concentric circles using as radii the distances from the center of the given circle to the points of intersection of the perpendiculars with the semicircumference. Prove this construction correct. Make construction for proportional parts.

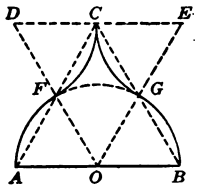


5. Construct the figure showing this arrangement of marble tiling in a floor. Find the cost of tiling for a corridor 15 ft. by 75 ft., if the white marble costs 35 cents per square foot and the colored 60 cents.



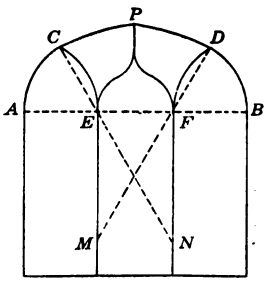
6. Construct the geometric figures in the following ornamental steel ceiling designs and explain. Notice that in both constructions, the arcs are drawn with reference to a square. How do they differ? Which has the centers of the arcs on the sides of a square? Which at the vertices of a square?

7. In this figure is a three centered ogee arch, in which $\triangle ACB$ is equilateral. Prove that arcs BG and CG are each $\frac{1}{6}$ of a circumference. Prove that C, G, and B are collinear, that is, in the same straight line. Prove that the area of the arch is equal to the area of the triangle ACB.



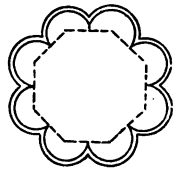
8. Draw a three centered ogee arch in which $OC = AB$. Find CE if $OC = AB = 10$ ft.

***9.** Construct the window of three lights as in the figure. The main arch is a four centered arch with the span divided into three equal parts. The centers are at E, F, M, and N. The two side arches are equilateral Gothic arches. The center one is a three centered ogee arch. Prove that $CN = DM = AB$.



10. In a regular hexagon, inscribe three equal circles each tangent to two sides of the hexagon and to the two other circles.

11. In a regular octagon, inscribe four equal circles, each tangent to two sides of the octagon and to two other circles.



12. The figure gives a suggestion for a doily pattern. Construct the pattern, making the octagonal center 8 in. across, and the radius of the outer arcs of the scallops $\frac{1}{4}$ in. longer than the radius of the inner arcs.

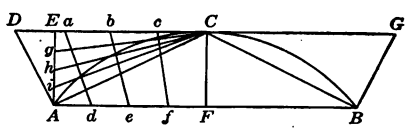
13. Two circles having diameters 2 in. and 3 in., respectively, are tangent externally. Draw a common secant to these circles such that the intercepted chords shall be 1 in. and 2 in., respectively.

14. Study this picture of the arches over the doorway of the Ducal Palace, Venice. Notice the ogee arches, Gothic arches, circles, trefoils, and quatrefoils. Construct the geometric outline, making the span of the outer arch 6 in.

15. Study this picture, which shows connected arches from the Golden Palace, Venice. Construct a portion of the geometric outline of the picture and explain.

16. Study the picture of the east end of Lincoln Cathedral, page 9, and make a geometric construction of the windows that are shown. Compare with Exercises 1 and 2, page 137.

***17.** The following construction is made to draw the arc of a segment when the chord and the height of the segment are given. This method is to be used when it is not convenient to find, or to use the radius. In the figure, AB is the chord and FC is the height. Draw AC; $AD \perp AC$; $CD \parallel FA$; $AE \perp AF$; and divide AF, DC, and AE into the same number of equal parts. Letter them as in the figure. Draw da, eb, fc, Cg, Ch, and Ci. The points of intersection of these give points on the arc. It is evident that, if the lines are divided into a greater number of equal parts, a larger number of points of the arc will be determined Give a proof of the construction.



COMPUTATIONS

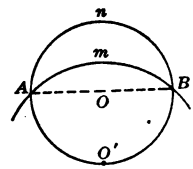
1. A furnace has 64 sq. ft. of grate surface. If the law allows 3 sq. ft. of grate surface to every square inch of safety valve, what diameter of safety valve should be used on a boiler having a furnace with the above grate surface?

2. Given that copper will stand a tensile strain of 30,000 pounds per square inch, find the greatest tensile pull a copper wire having a diameter of 0.25 in. can stand without breaking.

3. What is the diameter of a round cast-iron bar suspended from one end, which is subjected to a tension of 30,000 pounds, if the stress per square inch is 2400 pounds?

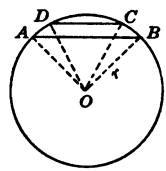
4. A canvass canopy of a hammock is 6 ft. by 10 ft., with a flounce hanging down 9 in. on all sides. The lower edge is scalloped in half-circular cuts, as shown in the figure. If each scallop is 4 in. across, find the number of yards of tape necessary to bind the flounce. Would it take any more tape if the scallops were 6 in. across?

5. Find the perimeter and the area of the crescent $AnBm$ formed by arcs of circles with centers at O and O' respectively, OB being 8 in.



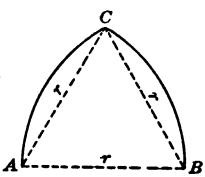
6. The sides AB of an inscribed square and DC of an inscribed regular hexagon are parallel; find the area included between them if the radius of the circle is 10 ft.

7. The inner and the outer diameters of a ring are $9\frac{1}{2}$ and 10 in. respectively; find the area of the ring.



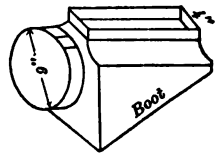
8. In putting up blower pipes, two circular pipes of diameters 11 in. and 14 in., respectively join and continue as a rectangular pipe 14 in. wide. Find the length of the cross section of the rectangular pipe.

9. The Gothic Arch shown in the figure is formed of two arcs, each one-sixth of a circle, the center of each of which is at the extremity of the span, *i.e.*, the radius equals the span. Find the area of such an arch of radius 5 ft.



10. How many times the area of the cross section of a $\frac{1}{16}$-inch wire is a $\frac{1}{2}$-inch wire?

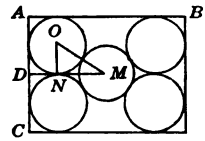
11. A hot air pipe of diameter 9 in. passes into a boot as shown in the figure, and a rectangular pipe of the same capacity passes upward from the boot. If the rectangular pipe is 4 in. wide, find its length in cross section.



12. If an arc of a circle is taken equal in length to the radius, find the value of the central angle measured by the arc.

13. The length of the circumference of a circle is 132 ft. Calculate the length of an arc of 40°, and the area of a sector of 80°.

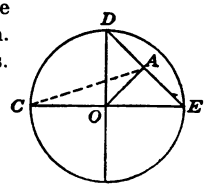
14. Four of the largest possible equal sized pipes are enclosed in a box of square cross section 18 in. on an edge. What part of the space do the pipes occupy?



***15.** Find size of the box to enclose five 6-inch pipes, placed as in the figure. What part of the area of the box is the area of the pipes?

16. Using 4000 miles as the radius of the earth, find the length in miles of an arc of 1′ (*a*) on the parallel of 45° north; (*b*) on the parallel of 60° north.

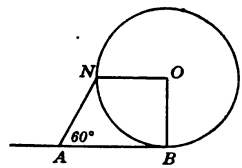
Ans. (*a*) 0.823− miles; (*b*) 0.582− miles.



17. What is the per cent of error in taking 4 times CA in the figure as the circumference of circle O?

Ans. 0.66+% too large.

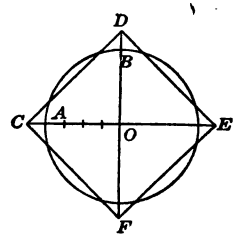
18. Make a construction as shown in the figure. AB is approximately the quadrant of the circle. Find the per cent it differs from the correct value. *Ans.* 0.4% too large.



19. Find the area of the sector in a circle whose radius is 28 cm., if the sector contains an angle of 50° 36′. *Ans.* 346.19 sq. cm.

20. Justify the following rule used by sheet metal workers, or show the per cent of error if it is not correct. Divide the radius AO into four equal parts, place one of these parts from A to C and another from B to D, the ends of two perpendicular diameters. Connect C and D, which gives the side of a square of the same area as the circle.

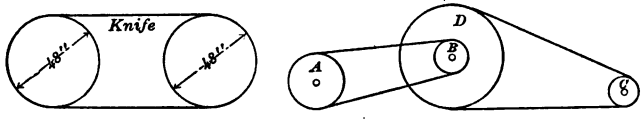
Ans. 0.5+% too small.



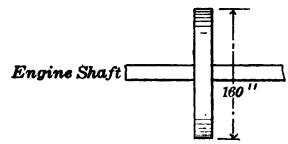
***21.** Find the area of a sector whose arc is 99.58 m. long and whose radius is 86.34 m. *Ans.* 4298.87 sq. m.

*22. Find the diameter of a circular shaft that may have a triangular end $2\frac{1}{2}$ in. on a side. A pentagonal end $1\frac{1}{2}$ in. on a side. An octagonal end $1\frac{1}{8}$ in. on a side. *Ans.* 2.887− in.; 2.552− in.; 2.940− in.

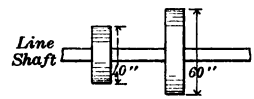
*23. An endless knife runs on pulleys having a diameter of 48 in. as shown in the figure and makes 180 R.P.M. (revolutions per minute). If the pulleys have their diameters decreased 18 in., how many R.P.M. will they have to make to keep the knife moving at the original speed?



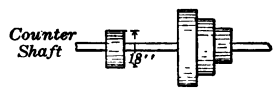
*24. Pulleys are arranged as in the figure. Pulley *A*, making 192 R.P.M., is the driver, and has a diameter of 14 in. Pulley *B* has a diameter of 8 in. Pulley *C* has a diameter of 6 in., and is to make 1400 R.P.M. Find the diameter of pulley *D*, fastened to the same shaft as *B*, in order that *C* may have the desired number of R.P.M. *Ans.* 25 in.



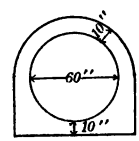
*25. In the figure, if a 160-in. pulley on the engine shaft drives a 60-in. pulley on the line shaft, and a 40-in. pulley on the line shaft drives an 18-in. pulley on the counter shaft, find the number of revolutions per minute of the counter shaft if the engine runs at 80 R.P.M.



26. If an error of 1% is made in measuring the radius of a circle, find the per cent of error in the area of the circle when determined from the incorrectly measured radius.



27. In the figure is shown the cross section of a concrete conduit. Find the area of the cross section in square feet.



28. What is the radius of a circle whose area is doubled when the radius is increased by 2 ft.?

29. Find the diameter of a circle if the area has as many square inches as the circumference has linear inches.

30. A grindstone when new has a diameter of 48 in. What is its diameter when $\frac{1}{4}$ ground away? When $\frac{1}{2}$ ground away? When $\frac{3}{4}$ ground away?

***31.** In two connected belt pulleys or gear wheels, if D is the diameter of the driving wheel, d the diameter of the driven wheel, R the number of R.P.M. of driver, and r the number of R.P.M. of the driven wheel, show that $r=\frac{RD}{d}$.

***32.** In any system of pulleys or gears, the general rule holds that the product of the diameters or numbers of teeth of the driving wheels and the number of revolutions per minute of the first driver must be equal to the product of the diameters or the numbers of teeth of the driven wheels and the number of revolutions per minute of the first driven wheel. As a formula this may be stated,

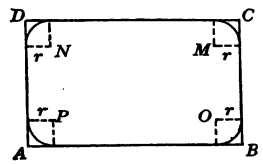
$$r=\frac{R\times D\times D'\times D''\times D'''\times \text{ etc.}}{d\times d'\times d''\times d'''\times \text{ etc.}}$$

where D, D', D'', etc. are the diameters of the driving pulleys, d, d', d'', etc. are the diameters of the driven pulleys, R is the R.P.M. of the first driver, and r is the R.P.M. of the last driven pulley. Show why this is true.

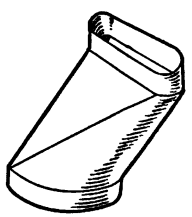
33. If the length of the equator were increased by 6 ft., how far from the surface of the earth would it be? Consider the equator as a perfect circle.

Ans. $\frac{3}{\pi}$ ft.

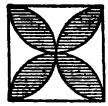
34. In laying out a running track, it was necessary to determine the radii of the quadrants to be formed at each corner of a rectangle 280 ft. by 180 ft. so that the track would take six laps for a mile. Find the length of the radii. *Ans.* $r=23.30$ ft.



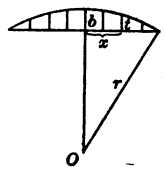
***35.** A practical rule (rule of thumb) for determining the length of an arc is $s=\frac{8c-C}{3}$, where s = length of arc, C = length of chord of arc, and c = length of chord of half of arc. Find the per cent of error in this rule for a 30° arc. For a 90° arc. Usually for an arc of more than 60° it is better to divide the arc into parts and compute the arc of each part.



36. A circular pipe of diameter 10 in. passes into a flattened pipe with $\frac{3}{4}$ as much area. If the flattened pipe has a cross section in the form of a rectangle 4 in. wide with a semicircle at each end, find the length of the cross section. *Ans.* 11.6 in.



37. If semicircles are described within a square whose side is 10 in. upon the four sides as diameters, find the area of the four lobes bounded by the eight quadrants.



***38.** When the arc of a segment of a circle is required for which the radius cannot be used, either by reason of its extreme length or because the center is inaccessible, it is desirable to obtain the arc without the use of the radius. This may be done by dividing the chord into a number of equal parts and calculating the several distances from the chord to the arc. Show that any distance $t = \sqrt{r^2 - x^2} - (r - b)$, where x is the distance from the center of the chord.

QUESTIONS

1. Give illustrations of cases where it would be best to find the area of a circle from the measurement of the circumference. From measurement of the diameter. Show why.

2. What measurements are necessary in order to determine completely a circle? A regular hexagon? A regular octagon?

3. What regular polygons can be constructed by aid of compasses and ruler?

4. How can circumferences be added? Subtracted? Multiplied? Divided? How can areas of circles be added? Subtracted? Multiplied? Divided?

5. How can the circumference be found from the radius? From the diameter? The diameter from the circumference? The area from the radius? From the diameter? The radius from the area? The diameter from the area? The circumference from the area?

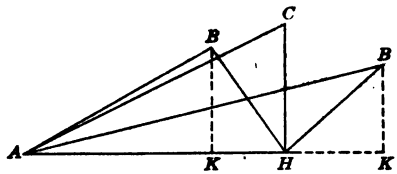
6. To what are the circumferences of circles proportional? The areas? The perimeters of regular polygons? The areas of regular polygons? The sectors of the same circle?

7. Give the sides of the following regular polygons in terms of the radius of the circumscribed circle: triangle, square, hexagon, decagon.

MAXIMA AND MINIMA

510. Definition. If several geometric magnitudes satisfy certain given conditions, the *greatest* is the **maximum** and the *smallest* is the **minimum.**

511. Theorem. *Of all triangles having two sides given, that in which these sides include a right angle is the maximum.*



Given rt.$\triangle ACH$ and $\triangle ABH$, having AH in common and $BH = CH$.

To prove $\triangle ACH > \triangle ABH$.

Proof. Draw the altitude BK.

	$BH > BK$.	Why?
Since	$BH = CH$,	Why?
Then	$CH > BK$.	
	$\therefore \triangle ACH > \triangle ABH$.	Why?

EXERCISES

1. What is the maximum of all chords that can be drawn in a given circle?

2. What is the minimum of all chords that can be drawn through a given point within a circle?

3. What is the maximum of all lines that can be drawn within a square?

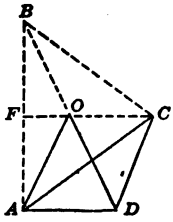
4. What is the minimum of all lines that can be drawn joining two parallel lines?

5. What is the maximum of all parallelograms which have two adjacent sides given?

6. What is the minimum of all lines that can be drawn to a circle from an outside point?

512. Isoperimetric Figures. Closed figures which have equal perimeters are called **isoperimetric figures.**

513. Theorem. *Of all triangles having the same base and altitude, the isosceles triangle has the least perimeter.*



Given $\triangle AOD$, an isosceles triangle, and $\triangle ACD$, any triangle not isosceles, having the same base and altitude.

To prove perimeter of $\triangle AOD <$ perimeter of $\triangle ACD$.

Proof. Draw $AB \perp AD$ and $CF \perp AB$.

Produce DO to meet AB at B and draw BC.

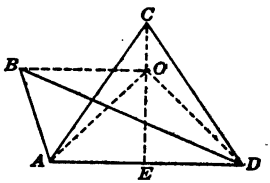
$AO = OB$, and $AC = CB$. Why?

$DO + OB < DC + CB$. Why?

Then $DO + OA < DC + CA$.

$\therefore$ perimeter of $\triangle AOD <$ perimeter of $\triangle ACD$.

514. Theorem. *Of all isoperimetric triangles having the same base, the isosceles triangle is the maximum.*



Given $\triangle ACD$, an isosceles triangle, and $\triangle ABD$, any triangle not isosceles, having the same base AD and equal perimeters.

To prove $\triangle ACD > \triangle ABD$.

Proof. Draw $CE \perp AD$ and $BO \parallel AD$.

Also draw AO and DO.

$\triangle AOD$ is isosceles. Why?

Perimeter of $\triangle ABD >$ perimeter of $\triangle AOD$. § 513

Perimeter of $\triangle ACD >$ perimeter of $\triangle AOD$.

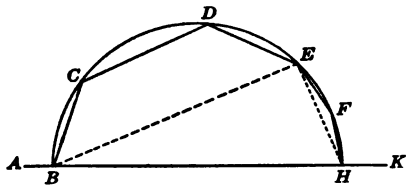
Then $AC > AO$ and $CE > OE$. Why?

$\triangle ACD > \triangle AOD$. Why?

$\triangle ABD = \triangle AOD$. Why?

$\therefore \triangle ACD > \triangle ABD$.

515. Theorem. *Of all polygons with sides all given but one, the maximum can be inscribed in a semicircle which has the undetermined side for a diameter.*



Given $BCDEFH$, the maximum of all polygons with sides BC, CD, DE, EF, and FH having the vertices B and H on the line AK.

To prove $BCDEFH$ can be inscribed in a semicircle whose diameter is BH.

Proof. Connect any vertex, as E, with B and H.

If $\angle BEH$ is not a right angle, the area of $\triangle BEH$ can be increased by moving B and H along AK, keeping the lengths of BE and EH unchanged, until $\angle BEH$ becomes a right angle while the area of the remainder of the polygon is unchanged. § 511

But, since it is given that the polygon is a maximum, its area cannot be increased.

Then $\angle BEH$ is a right angle, and the semicircle described on BH as a diameter will pass through E. Why?

In like manner, it may be proved that the semicircle passes through all the vertices of the polygon.

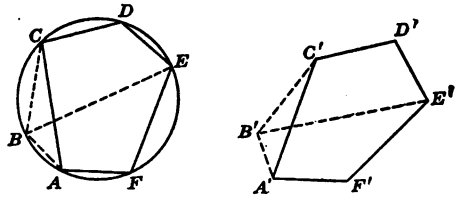
$\therefore$ the polygon $BCDEFH$ may be inscribed in a semicircle with BH as a diameter.

EXERCISES

1. What is the minimum of all circles which can be described on a given line as a chord?

2. Of all rectangles having a given area, that which has the least perimeter is the square.

516. Theorem. *Of all polygons having their sides respectively equal, the maximum can be inscribed in a circle.*



Given polygon $ACDEF$ which can be inscribed in a circle, and polygon $A'C'D'E'F'$ which cannot be inscribed in a circle, and that the sides of the two polygons are respectively equal.

To prove $ACDEF > A'C'D'E'F'$.

Proof. Draw the diameter BE and the chords BA and BC. On $A'C'$, construct $\triangle A'B'C' = \triangle ABC$ and draw $B'E'$.

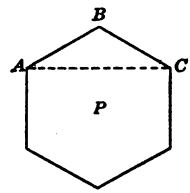
$BCDE > B'C'D'E'$, and $BAFE > B'A'F'E'$. § 515

Then $ABCDEF > A'B'C'D'E'F'$. Why?

But $\triangle ABC = \triangle A'B'C'$. Why?

$\therefore ACDEF > A'C'D'E'F'$. § 174

517. Theorem. *Of all isoperimetric polygons having the same number of sides, the maximum is a regular polygon.*



Given P, the maximum polygon of a given number of sides.

To prove that P is a regular polygon.

Proof. Draw the diagonal AC

$\triangle ABC$ is the maximum of isoperimetric triangles on the base AC, since P is a maximum polygon.

Then $AB = BC$. § 514

In like manner, all the sides can be proved equal.

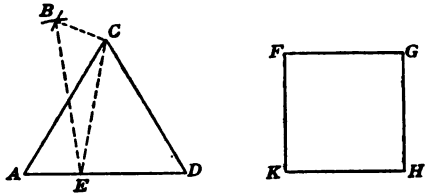
Hence P is an equilateral polygon. Why?

$\therefore P$ is a regular polygon. §§ 516, 459

EXERCISE

Inscribe the maximum rectangle in a circle.

518. Theorem. *Of all isoperimetric regular polygons, the maximum has the greatest number of sides.*



Given the two isoperimetric regular polygons ACD and $FGHK$, having three and four sides respectively.

To prove $\square FGHK > \triangle ACD$.

Proof. Draw CE from C to any point in AD. Construct $\triangle CEB = \triangle CEA$, with $BE = AC$ and $BC = AE$.

Then the quadrilateral $BCDE$ has the same area and the same perimeter as $\triangle ACD$.

But square $FGHK >$ quadrilateral $BCDE$. § 517

$\therefore \square FGHK > \triangle ACD$.

In like manner, it can be shown that a regular pentagon is greater than a square having the same perimeter, and so on indefinitely.

519. Theorem. *The area of a circle is greater than the area of a regular polygon having the same perimeter.*

This must follow, as in § 518, when the number of sides of regular polygons has been increased indefinitely.

EXERCISES

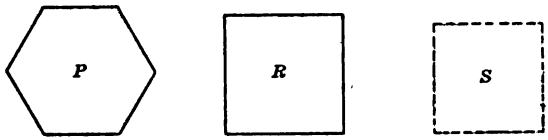
1. The rectangle is the maximum of all parallelograms having given sides.

2. Divide a line into two parts so that the product of the two parts shall be a maximum.

3. Show how to inscribe the maximum rectangle in a semicircle.

4. Show how to inscribe an angle in a semicircle so that the sum of its sides shall be a maximum.

520. Theorem. *Of all regular polygons having the same area, the one which has the greatest number of sides has the least perimeter.*



Given P and R, regular polygons with the same area, P having more sides than R.

To prove perimeter of $P <$ perimeter of R.

Proof. Construct the regular polygon S, having the same perimeter as P and the same number of sides as R.

	$S < P$.	§ 518
But	$R = P$.	Why?
Then	$S < R$.	Why?
	Perimeter of $S <$ perimeter of R.	
But	perimeter of $S =$ perimeter of P.	Cons.
	$\therefore$ perimeter of $P <$ perimeter of R.	Why?

521. When the number of sides of the polygon P in § 520 has been increased indefinitely, it must follow that *the circumference of a circle is less than the perimeter of any regular polygon having the same area.*

EXERCISES

1. A square and a circle each have an area of 36 sq. in. Which has the greater perimeter? § 521

Find the perimeter of the square and the circumference of the circle.

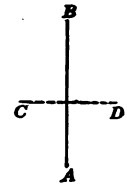
2. Compare the area and the perimeter of a lot 60 ft. square with the area and perimeter of a lot 90 ft. by 40 ft.

3. Compare the area and the perimeter of a lot 60 ft. square with the area and the perimeter of a lot 50 ft. by 70 ft.

4. The perimeter of the bases of two cans is the same. The base of one is a square and of the other a circle. Which one has the greater area of base? § 521

SYMMETRY

522. Two points are **symmetrical with respect to a straight line** as an axis if the straight line joining the two points is bisected at right angles by the axis.

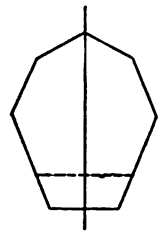


The points C and D are symmetrical with respect to AB as an axis, since AB is the perpendicular bisector of CD.

523. A figure is **symmetrical with respect to an axis** if every point on one side of the axis has a corresponding symmetrical point on the other side of the axis.

When this is true, the two parts of the figure will coincide if one part is folded over the axis.

The outline of the human form is symmetrical with respect to an axis.

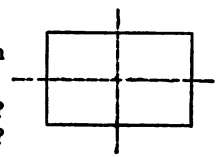


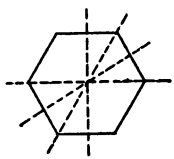
524. Axis of Symmetry. A line in a figure is an **axis of symmetry** if one part of the figure when folded over it coincides with the other part.

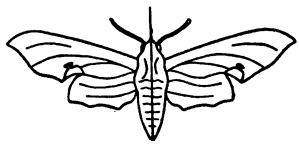
EXERCISES

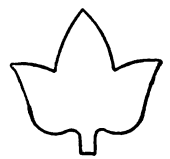
1. How many axes of symmetry has an isosceles triangle? An equilateral triangle?

2. How many axes of symmetry has a square? A rectangle? A rhombus? An isosceles trapezoid?







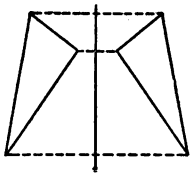


3. How many axes of symmetry has a regular pentagon? A regular hexagon?

4. Point out axes of symmetry in the outline figures given.

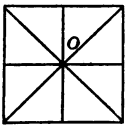
5. How many axes of symmetry has a circle?

525. Two figures are **symmetrical with respect to an axis** if every point in one figure has a corresponding symmetrical point in the other.



526. Center of Symmetry. If a point within a polygon bisects all lines which pass through it and are terminated by the perimeter, the point is a **center of symmetry.**

Thus O is the center of symmetry of the square.

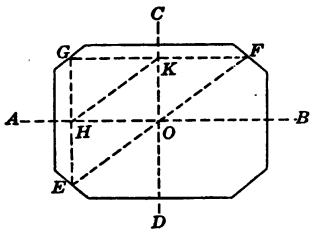


EXERCISE

Has an equilateral triangle a center of symmetry? A regular hexagon? A regular pentagon? A rectangle? A circle?

527. Theorem. *If a figure has two axes of symmetry perpendicular to each other, it is symmetrical with respect to their point of intersection as a center of symmetry.*

The polygon of the figure has two axes, AB and CD, perpendicular to each other, which intersect at O.



To prove that O is the center of symmetry of the polygon.

Proof. From any point G on the perimeter, draw $GF \perp CD$ and $GE \perp AB$.

Draw HK, EO, and OF.

Then $GE \parallel CD$ and $GF \parallel AB$. Why?

$GH = KO$ and $GH = HE$. Why?

$HE = KO$ and $HKOE$ is a parallelogram. Why?

Then $HK \parallel EO$ and $HK = EO$. Why?

In like manner, it can be shown that $HK \parallel OF$ and $HK = OF$.

Then EOF is a straight line and O is its middle point.

Since G is any point on the perimeter, O bisects all straight lines passing through O and terminated by the perimeter.

$\therefore$ O is the center of symmetry of the polygon. § 526

FORMULAS FOR REFERENCE

Rectangle, parallelogram. $A = bh$. $b = A \div h$. $h = A \div b$.

Triangle. $A = \frac{1}{2}bh$. $b = 2A \div h$. $h = 2A \div b$.

Trapezoid. $A = \frac{1}{2}h(b+b')$.

Right triangle. $c = \sqrt{a^2+b^2}$. $a = \sqrt{c^2-b^2}$. $b = \sqrt{c^2-a^2}$.

Diagonal of a square = side $\times \sqrt{2}$. $d = s\sqrt{2}$.

Side of a square = $\frac{1}{2}$ diagonal $\times \sqrt{2}$. $s = \frac{1}{2}d\sqrt{2}$.

Altitude of equilateral triangle = $\frac{1}{2}$side $\times \sqrt{3}$. $h = \frac{1}{2}s\sqrt{3}$.

Area of square = $(\text{side})^2 = \frac{1}{2}(\text{diagonal})^2$. $A = s^2 = \frac{1}{2}d^2$.

Area of equilateral triangle = $\frac{1}{4}(\text{side})^2\sqrt{3}$. $A = \frac{1}{4}s^2\sqrt{3}$.

Any triangle. $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

Inscribed equilateral triangle. Side $= r\sqrt{3}$

Circumscribed equilateral triangle. Side $= 2r\sqrt{3}$.

Inscribed square. Side $= r\sqrt{2}$.

Circumscribed square. Side $= 2r$.

Inscribed regular pentagon. Side $= \frac{1}{2}r\sqrt{10-2\sqrt{5}} = 1.1756r$.

Inscribed regular hexagon. Side $= r$.

Circumscribed regular hexagon. Side $= \frac{2}{3}r\sqrt{3}$.

Inscribed regular octagon. Side $= r\sqrt{2-\sqrt{2}} = 0.7653r$.

Circumscribed regular octagon. Side $= 2r(\sqrt{2}-1) = 0.8284r$.

Inscribed regular decagon. Side $= \frac{1}{2}r(\sqrt{5}-1) = 0.618r$.

Area of regular polygon. $S = \frac{1}{2}ap$. § 478

Circumference. $c = \pi d = 2\pi r$.

Diameter. $d = c \div \pi = \sqrt{A \div \frac{1}{4}\pi}$.

Radius. $r = c \div 2\pi = \sqrt{A \div \pi}$.

Length of arc. $s = \frac{u}{360} \cdot \pi d$. § 496

Area of circle. $A = \frac{1}{2}cr = \pi r^2 = \frac{1}{4}\pi d^2$.

Area of ring. $A = \pi R^2 - \pi r^2 = \pi(R+r)(R-r)$. (Ex. 8, p. 249).

Area of sector. $S = \frac{1}{2}rs$. § 508

Area of segment of circle. $G = S - \frac{1}{2}r^2 \sin u$, § 509

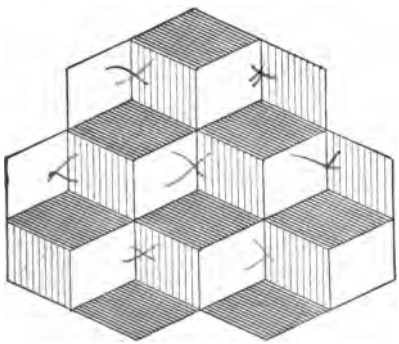
SOLID GEOMETRY

INTRODUCTION

528. Solid geometry may be thought of as an extension of plane geometry, with which it has much in common. The figures of plane geometry lie in a plane; while in solid geometry we consider the properties of figures that do not lie entirely in one plane. Such figures are sometimes called **three-dimensional figures.**

529. Space Ideas. We gain ideas concerning space and objects in space through the senses of touch, sight, and hearing. Mainly through touch and sight, we determine the size and shape of an object.

530. Difficulties. Most beginners in the study of solid geometry have difficulty in visualizing, or forming a mental image, of a three-dimensional figure from a worded description or from a drawing in a plane.



Even the same picture may present to the minds of two persons quite different images. For instance, in the figure, one person may see six blocks while another will see seven.

A little practice, however, in visualizing the blocks enables one at will to see six or seven blocks.

Frequently a student does not understand a proposition or a proof because he does not visualize the figure properly. Lines may appear to extend in a different direction than was intended or a plane may seem to lie in front of the page when it was intended that it should appear back of the page.

Whatever device may be necessary for a clear image should be used; but care should be taken not to become too dependent upon models. A very important part of the training from the study of solid geometry is that it trains in visualizing three-dimensional figures from a description or a drawing.

531. Devices to Assist in Imaging. Cardboard models may be formed as described in § 753.

Planes can be made to intersect by cutting two pieces of cardboard half in two and fitting together along the cuts.

Cardboard can be used in connection with sharp pointed wires, such as hatpins. § 568.

Hatpins or sharp wires held together by small corks can be used to form open models. § 571.

Various figures can be formed with the hatpins and a piece of soft board.

532. Representation of Solid Geometry Figures. In solid geometry, the figures are usually represented on paper or on the blackboard, that is, in a plane. It is well to have some idea of how to make the figures appear solid, and to "stand out" as desired. While it is not thought best to enter into a discussion of perspective, a few general suggestions that are of assistance will be given.

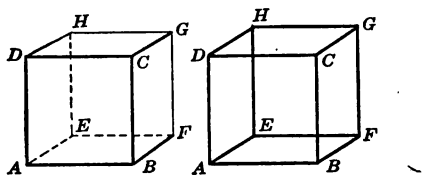
The figures may be supposed to be viewed from a point a little above, and either directly in front or a little to the right or left. Parts of the figure that are rectangles, in general, appear as parallelograms in the drawing. §§ 663-5, 671, 678. Polygons appear as polygons, but reduced in one direc-

tion. §§ 660, 749. Parts that are circles appear as ellipses, which become narrower as the eye of the observer is nearer to the plane in which the circle lies. §§ 689, 692, 864.

Planes are supposed to be opaque, and all lines covered by them should be left out, or, if it is desired to insert them, they should be dashed lines.

Two lines that intersect in the drawing do not necessarily intersect in the solid figure. Lines are not usually of the same length as they are drawn. The size of an angle may be quite different from the size in the drawing.

In the accompanying drawings representing cubes, the one at the left appears to have the face *ABCD* in front and can hardly be imaged otherwise. While the one at the right can easily be seen as if viewed from a point a little above and to the right, and so having the face *ABCD* in front; or it may be visualized as viewed from a point a little below and to the left, and so having the face *EFGH* in front.



533. The Use of Plane Geometry in Solid Geometry. All of the plane geometry is at the disposal of the student in solid geometry; but, in applying the facts of plane geometry, great care must be taken not to use them except where they will hold.

Facts of plane geometry that hold without reference to any plane are the axioms and most definitions, theorems stating that figures are congruent, and all the theorems where it is evident that it does not matter if the figures are not in the same plane.

Theorems of plane geometry that hold because the nature of the figure requires that it lie in a plane are those concern-

ing a triangle, two parallel lines, a parallelogram, a circle, and all other theorems where the conditions stated determine a plane in which the parts must lie.

In all other cases, the theorems of plane geometry can be applied only after all the parts concerned are shown to lie in one plane. Under these will fall many theorems concerning parallels, perpendiculars, circles, and polygons.

EXERCISES

1. Draw the figure of § 603, omitting the lines DE, EF, and DF. Can you visualize the figure as otherwise than in one plane?

2. Draw the figure of § 572, making the side lines of plane P full. Compare with the figure of the text.

3. Draw the figure of § 628, making all lines full and of the same thickness. Compare with the figure of the text.

4. Draw the figure of § 671, making all lines of the same thickness. Does your figure appear as if viewed from below? If not, make such lines heavy as will make it appear so.

5. Draw a figure like the one at the right in § 722, but change the lines so that it will appear as if viewed from below.

6. Draw a figure of a sphere as in § 787, but viewed from a point on a level with the center of the sphere.

CHAPTER VI

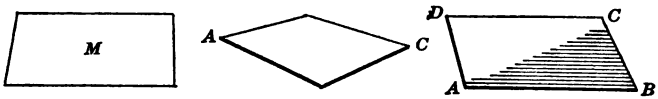
STRAIGHT LINES AND PLANES

534. Surface. A boundary of a portion of space is called a **surface.** A surface may be limited or unlimited in extent. It has two dimensions.

535. Plane. A surface such that a straight line joining any two of its points, lies wholly in the surface, is called a **plane surface** or, simply, a **plane.**

Since a straight line may be unlimited in length, this definition implies that a plane may be unlimited in extent, for otherwise the straight line could not lie wholly in the surface.

It also follows from the definition that a straight line can intersect a plane in but one point.

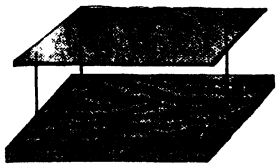


536. Representing a Plane. In drawing a geometric figure a portion of the plane is conveniently represented as a rectangle seen obliquely. In order to make the plane appear to "stand out," the front edge is frequently drawn longer than the back edge. Making the edges on the front and one end heavy and shading help one to image a plane as it is intended it should appear.

537. Reading a Plane. A plane may be read by a single letter, by letters at opposite vertices of the rectangle that represents it, or by the letters at all the vertices of the rectangle. Thus, in the figure, the planes may be read "plane *M*," "plane *AC*," or "plane *ABCD*."

DETERMINING A PLANE

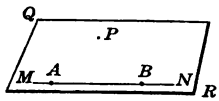
538. A plane is said to be *determined* by given conditions if that plane and no other plane fulfills those conditions.



539. Axiom. *Three points not in a straight line determine a plane.*

That is, one plane and only one plane contains three points that are not in a straight line.

540. Theorem. *A straight line and a point not in that line determine a plane.*



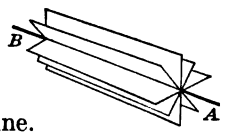
Given the straight line MN and the point P not in the line.

To prove that MN and P determine a plane.

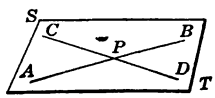
Proof. Let A and B be any two points in line MN.

Then A, B, and P determine a plane.	§ 539
Also MN lies in that plane.	§ 535
$\therefore$ MN and P determine a plane.	§ 538

541. Remark. Since a straight line and any point in space determine a plane, it follows that an indefinite number of planes can be passed through a straight line.



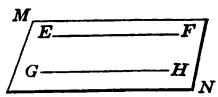
542. Theorem. *Two intersecting straight lines determine a plane.*



One line and a point in the other determine a plane.

Complete the proof.

543. Theorem. *Two parallel lines determine a plane.*

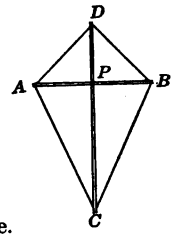


544. Theorem. *A straight line lies in a plane if it has a point in the plane and is parallel to a line in the plane.*

EXERCISES

1. Hold two pencils so that they determine a plane. So that they do not determine a plane.

2. In making a kite, two straight sticks AB and CD are fastened together at P. Show that paper stretched over these sticks lies in a plane.



3. Show that a rubber cord fastened at both ends and stretched by grasping it at any other point, lies in one plane.

4. From a given point lines are drawn to points on a given line; show that all these lines lie in the same plane.

5. What is the locus of all lines that pass through a given point and intersect a given line not containing the point?

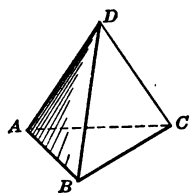
6. Show that, if a rubber band is stretched by three hooks in any position, all parts of the band lie in one plane.

7. Do four points necessarily lie in a plane? When do they? Give illustrations by using points in the classroom.

8. Why is a tripod used to support a camera or a surveyor's transit?

9. Why do the four legs of a chair sometimes not all rest upon a floor?

10. How many planes are determined by four points not all lying in the same plane?



11. Given five points four of which are in the same plane. How many planes do they determine?

12. How many planes are determined by three concurrent lines that do not all lie in the same plane?

13. How does a carpenter determine whether a floor is a plane?

14. Why does a mason use a trowel with long straight edges when "truing up" a wall?

15. How many planes are determined by four lines all meeting in a point but no three of which lie in the same plane?

16. Must a triangle lie in a plane? Must a parallelogram? Must a trapezoid? Must every quadrilateral?

17. Is the following a complete definition for a circle: A curved line every point of which is equally distant from a fixed point called the center is a circle? Why?

18. Through a point in space (1) draw a line parallel to a given line; (2) draw a line perpendicular to a given line.

RELATIVE POSITIONS OF LINES AND PLANES

545. Definitions. **Collinear** means lying in the same line. **Coplanar** means lying in the same plane.

546. Relations of Two Lines. From the study of plane geometry and §§ 542, 543, it follows:

(1) That two coplanar lines may *coincide, intersect,* or *be parallel.*

(2) That two non-coplanar lines are *neither intersecting nor parallel.*

547. Relations of a Line and a Plane. A straight line may have three positions relative to a plane:

(1) *It may lie in the plane.*

(2) *It may intersect the plane.*

(3) *It may be parallel to the plane.*

A line **lies in a plane** if all of its points are in common with the plane.

A line **intersects a plane** if it has only one point in common with the plane. The point in common is called the **foot** of the line.

A line is **parallel to a plane,** and the plane is parallel to the line, if they have no point in common, that is, if they do not meet.

548. Relations of Two Planes. Two planes may have three relative positions:

(1) *They may coincide.*

(2) *They may intersect.*

(3) *They may be parallel.*

Two planes are said to **coincide** if they have the same determining conditions.

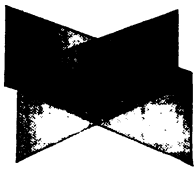
In plane geometry, the relative positions of points and lines are few and so the treatment of the subject is quite informal. In solid geometry, the study is made of points, lines, and planes, and their relative positions are many. It is advisable then, to make a more formal study of the subject.

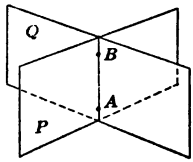
INTERSECTING PLANES

549. Definition. The **intersection of two surfaces** consists of all points common to the two surfaces, and no other points; that is, it is the locus of all points common to the two surfaces.

550. Axiom. *If two planes have one point in common, they also have another point in common.*

551. Theorem. ***If two planes intersect, their intersection is a straight line.***





Given two intersecting planes P and Q.

To prove that their intersection is a straight line.

Proof. Let A and B be two points common to both planes. § 550

The straight line determined by A and B lies in the plane P and also in Q. § 535

Then AB is common to the two planes. Why?

Moreover, no point not in AB is common to the two planes for then they would coincide. § 540

$\therefore$ the intersection of planes P and Q is a straight line.

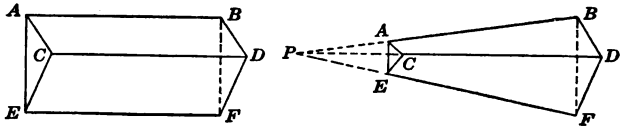
EXERCISES

1. Can three planes intersect (1) in one line? (2) in two lines? (3) in three lines? (4) in more than three lines? Explain.

2. If a paper is folded and creased, why does it form a straightedge along the crease?

3. From a point external to two non-coplanar lines only one line can be drawn that will cut the two lines.

552. Theorem. *If three planes, not passing through the same line, intersect each other, their three lines of intersection either meet in a point or are parallel each to each.*



Given the three planes AD, CF, and AF not passing through the same line, and intersecting in lines AB, CD, and EF.

To prove that AB, CD, and EF either meet in a point or are parallel each to each.

Proof. Consider the two lines of intersection lying in any plane. (1) These lines intersect, or (2) they are parallel.

(1) Suppose AB and CD intersect at P.

Then P lies in plane AF and in plane CF, and hence in their intersection EF. § 549

$\therefore$ AB, CD, and EF meet in point P.

(2) Suppose $AB \parallel CD$.

Then $EF \parallel CD$ or intersects CD.

Suppose EF intersects CD, then AB intersects CD by (1).

But this is impossible, for $AB \parallel CD$.

Hence $EF \parallel CD$.

Similarly it can be proved that $EF \parallel AB$.

$\therefore$ AB, CD, and EF are parallel each to each.

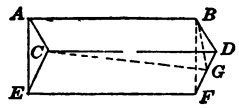
EXERCISES

1. Three planes that do not contain the same straight line can have but one point in common.

2. Explain the meaning of the expression, "Two planes determine a straight line." Is it always true that two planes determine a straight line? Is the analogous statement, "Two lines determine a point," of plane geometry always true? Explain.

LINES AND PLANES, PARALLEL

553. Theorem. *Two straight lines that are parallel to a third straight line are parallel to each other.*



Given lines AB and CD, each parallel to EF.

To prove $AB \parallel CD$.

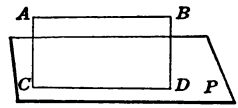
Proof. Either AB is parallel to CD or it is not.

	AB and EF determine plane AF.	Why?
	CD and EF determine plane CF.	Why?
	AB and C determine plane BC.	Why?

If AB is not parallel to CD, plane BC will intersect plane CF in some line CG other than CD. § 551

Then	$CG \parallel EF$.	§ 552 (2)
But	$CD \parallel EF$.	Given
Hence	CD and CG coincide.	§ 118
	$\therefore AB \parallel CD$.	Why?

554. Theorem. *A plane containing one, and but one, of two parallel lines is parallel to the other.*



Given two parallel lines AB and CD, and plane P containing CD but not AB.

To prove plane $P \parallel AB$.

Suggestion. AB and CD determine a plane. Suppose AB not parallel to plane P, and show that then AB would not be parallel to CD.

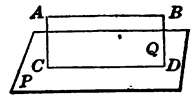
EXERCISES

1. As a door is opened, show that all the positions of its outer edge are parallel to each other.

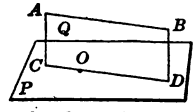
2. Prove that the lines connecting in order the middle points of the sides of a quadrilateral in space form a parallelogram.

By a "quadrilateral in space" is meant a quadrilateral not all of whose sides lie in the same plane. It is often called a **skew** or **gauche quadrilateral.**

555. Theorem. *If a line is parallel to a plane, it is parallel to the intersection of that plane with any plane through the line.*



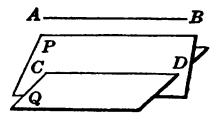
556. Theorem. *A line is in a plane if it has a point in the plane, and the line and the plane are both parallel to a second line.*



The two parallel lines determine a plane intersecting the first plane.

557. Theorem. *If a line is parallel to the intersection of two planes, it is parallel to each of the planes.*

558. Theorem. *A line parallel to each of two intersecting planes is parallel to their intersection.*



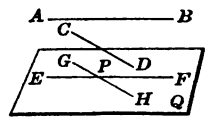
A line through a point in the intersection of the two planes and parallel to the given line must lie in both planes by § 556.

559. Problem. *Through a given line to pass a plane parallel to any other straight line not intersecting the given line.*

(1) How many planes can be passed through the given line and parallel to the other line if the given line is parallel to the other line? §§ 541, 554.

(2) If the given line is not parallel to the other, construct a line parallel to the given line and through a point in the other. Then the intersecting lines determine a plane parallel to the given line.

560. Problem. *Through a point to pass a plane parallel to each of two given non-coplanar lines.*



Given AB and CD, the two lines, and P, the point.

Construction. Through P construct lines parallel to both AB and CD. These lines determine the plane parallel to AB and CD.

Give proof.

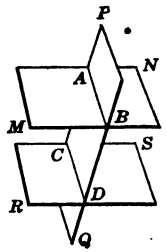
EXERCISES

1. If a line in one of two intersecting planes is parallel to the other plane, it is parallel to their intersection.

2. If a line and a plane are parallel to the same line, they are parallel to each other, or the line lies in the plane.

561. Definition. Two planes are parallel if they have no points in common, that is, if they do not meet however far they may be produced.

562. Theorem. *If a plane intersects two parallel planes, the lines of intersection are parallel.*



Given plane PQ intersecting the two parallel planes MN and RS in AB and CD, respectively.

To prove $AB \parallel CD$.

Suggestion. Since AB and CD are in the same plane PQ, they are parallel or intersect. Suppose that they intersect. What follows?

563. Theorem. *Parallel lines intercepted between parallel planes are equal.* § 158

564. Theorem. *A line is in a plane if it has a point in the plane and if the line and plane are both parallel to a second plane.*

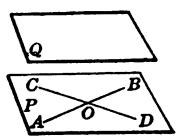
Pass a plane through the line and intersecting the two planes. Then apply § 118.

EXERCISES

1. If a line and a plane are parallel to the same plane, they are parallel to each other, or the line lies in the plane.
2. State the converse of the theorem of § 562. Is it true?
3. A board is cut across on a slant. What is the shape of the cut section?

565. Theorem. *If two intersecting straight lines are parallel to a plane, the plane determined by these lines is parallel to that plane.*

Given the intersecting lines AB and CD, each parallel to plane Q, and the plane P determined by AB and CD.



To prove $P \parallel Q$.

Proof. Either P is or is not parallel to Q.

If P is not parallel to Q, it will intersect it in a line parallel to both AB and CD. § 555

But this is impossible. § 118

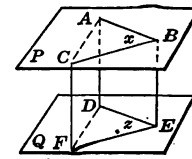
$\therefore P \parallel Q$.

566. Theorem. *Through a given point one plane, and only one can be passed parallel to any given plane not containing the point.*

Pass a plane through the given point and intersecting the given plane. Through the point and in this plane draw a line parallel to the intersection of the two planes. This line is parallel to the given plane by § 554.

*In a similar manner draw a second line through the given point, and apply § 565.

567. Theorem. *If two angles, not in the same plane, have their sides parallel, right side to right side and left side to left side, the angles are equal and their planes are parallel.*



Given $\angle x$ in plane P and $\angle z$ in plane Q, having $AB \parallel DE$ and $CB \parallel FE$.

To prove $\angle x = \angle z$, and $P \parallel Q$.

Proof. Take $BA = ED$, and $BC = EF$, and draw lines AD, CF, BE, AC, and DF.

Both AE and CE are parallelograms.		Why?
Hence both AD and CF are equal and parallel to BE.		Why?
Then	AD and CF are equal and parallel.	Why?
Therefore	$ADFC$ is a parallelogram.	§ 160
And	$AC = DF$.	Why?
Then	$\triangle ABC \cong \triangle DEF$.	§ 113
	$\therefore \angle x = \angle z$.	Why?
Also	$AB \parallel Q$, and $CB \parallel Q$.	§ 554
	$\therefore P \parallel Q$.	§ 565

EXERCISES

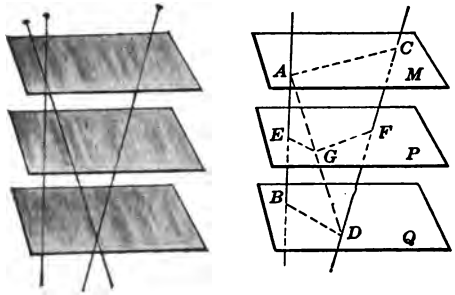
1. What is the locus of a line that passes through a given point, and is parallel to a given plane?

2. If two congruent parallelograms lie in parallel planes and have their sides respectively parallel, how many planes are determined in using their sides in all possible combinations?

3. The four lines in which two parallel planes intersect two other parallel planes are parallel.

4. If two planes are parallel to the same line, their intersections with any plane through the line are parallel to each other.

568. Theorem. ***If two straight lines are cut by three parallel planes, their corresponding segments are proportional.***



Given AB and CD two straight lines intersected by three parallel planes M, P, and Q, in the points A, E, and B, and C, F, and D respectively.

To prove $\frac{AE}{EB}=\frac{CF}{FD}$.

Proof. Draw AD intersecting plane P in G.

Let the plane determined by AB and AD intersect plane P in EG and plane Q in BD.

Then $EG \parallel BD$. § 562

Similarly, if AD and CD determine a plane intersecting plane M in AC and plane P in GF, $GF \parallel AC$. Why?

Hence, $\frac{AE}{EB}=\frac{AG}{GD}$, and $\frac{CF}{FD}=\frac{AG}{GD}$. § 411

$\therefore \frac{AE}{EB}=\frac{CF}{FD}$. Why?

569. Theorem. *If two straight lines are cut by a series of parallel planes, the corresponding segments are proportional.*

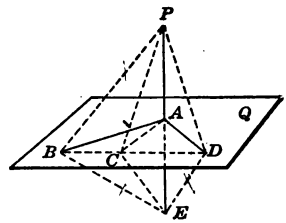
EXERCISE. A mine shaft passes for 180 ft. on a slant through two layers of rock. One, a layer of hard rock, is 72 ft. thick; and the other, a layer of soft rock, is 84 ft. thick. If it costs \$4 per linear foot in the hard rock and \$3 in the soft to make the shaft, find the cost of digging the shaft.

LINES AND PLANES, PERPENDICULAR

570. Definition. A straight line is **perpendicular to a plane** if perpendicular to all straight lines in the plane and passing through its foot. The plane is also **perpendicular to** the line.

A straight line is **oblique to a plane** when it meets the plane but is not perpendicular to it.

571. Theorem. ***If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by these lines.***



Given $AP \perp AB$ and AD at their point of intersection A, and plane Q determined by AB and AD.

To prove AP perpendicular to the plane Q.

Proof. In plane Q draw BD, any line not passing through A, and through A draw any line AC meeting BD in C.

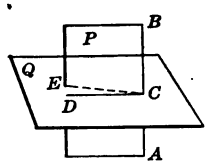
Prolong PA to E, making $AE = AP$; and draw BP, CP, DP, BE, CE, and DE.

Then	AB and AD are perpendicular bisectors of PE.	Why?
Therefore	$BP = BE$, and $DP = DE$.	§ 205
Then	$\triangle BPD \cong \triangle BED$.	§ 113
And	$\angle CBP = \angle CBE$.	Why?
Further	$\triangle CBP \cong \triangle CBE$.	§ 247
And therefore	$CP = CE$.	Why?

Hence $AC \perp PE$, *i.e.*, $AP \perp AC$, any line in plane Q and passing through its foot. Why?

$\therefore$ AP is perpendicular to the plane Q. § 570

572. Theorem. *All the perpendiculars that can be drawn to a straight line at a given point in the line, lie in a plane perpendicular to the line at the given point.*



Given DC any line $\perp AB$ at C, and plane $Q \perp AB$ at C.

To prove that all lines perpendicular to AB at C lie in plane Q.

Proof. DC and AB determine a plane P that intersects plane Q in EC.

Then	$EC \perp AB$.	§ 570
But	$DC \perp AB$.	Given
Therefore DC coincides with EC.		Why?
And DC lies in plane Q.		

$\therefore$ all lines $\perp AB$ at C lie in plane Q.

573. Theorem. *Through a given point one plane can be passed perpendicular to a given straight line, and only one.*

If the given point is in the given line, two perpendiculars to the line determine a plane perpendicular to the given line. If the given point is outside the given line, construct a perpendicular from the point to the line, and at its foot construct another perpendicular to the given line. These determine the perpendicular plane.

To show that only one plane can be drawn in each case, suppose that there are two planes perpendicular and pass a plane through the given line and intersecting the two planes in two lines. What follows?

EXERCISES

1. Show how a perpendicular to a plane can be determined by means of two carpenter's squares.

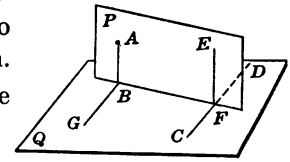
2. Each of three concurrent lines is perpendicular to each of the other two. Prove that each is perpendicular to the plane of the other two.

3. If a book partly open stands on a table, is the back of the book perpendicular to the table top?

4. One side of a right angle revolved about the other side as an axis generates a plane perpendicular to this other side.

5. Do the hands of a clock revolve in a plane? Why?

574. Theorem. *From a given point outside a plane one line perpendicular to the plane, and only one, can be drawn.*



Given plane Q and point A outside Q.

To prove one line perpendicular to Q from A, and only one, can be drawn.

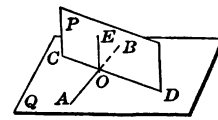
Proof. In plane Q draw any line CD. Through A pass a plane $P \perp CD$. § 573

Let BF be the intersection of P with Q.

Construct $AB \perp BF$, and $EF \parallel AB$.		§§ 312, 117
In Q, construct $GB \parallel CF$.		§ 117
Then	$CF \perp EF$.	Why?
And	$\angle ABG = \angle EFC$.	§ 567
Hence	$AB \perp GB$.	Why?
Therefore	$AB \perp$ plane Q.	§ 571

Only one perpendicular can be drawn, for if two could be drawn, they would both be perpendicular to the intersection of their plane with Q, which is impossible. Why?

575. Theorem. *At a point in a plane one perpendicular can be drawn to the plane, and only one.*



Let O be the point in plane Q. Draw AB, any line in Q through O. Construct plane $P \perp AB$ through O. In P construct $OE \perp CD$. Complete and give proof.

576. Theorem. *The perpendicular to a plane is the shortest line from a point to a plane.*

Apply § 185.

577. Definition. The length of the perpendicular from a point to a plane is called the **distance from the point to the plane.**

The **distance between two parallel planes** is the length of the segment of a common perpendicular lying between them

EXERCISES

1. What is the locus of all points equidistant from a circle?

2. What is the locus of all points equidistant from the vertices of a triangle?

3. What is the locus of all points equidistant from two given points?

4. Find the length of the locus in a plane of all points 10 in. from a point 6 in. from the plane.

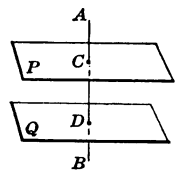
5. With a pair of compasses opened 13 in. show how a circle of 12 in. radius can be drawn.

6. A perpendicular smokestack is braced by three guy wires attached at a point 90 ft. from the ground. If these wires are 150 ft. long and meet the ground at the vertices of an equilateral triangle, find the length of one side of the triangle.

7. Could a smokestack be fully braced with three guy wires? With two? Why?

8. If from the foot of a perpendicular to a plane a line is drawn perpendicular to any given line in the plane, the line joining the point of intersection to any point in the perpendicular to the plane is perpendicular to the given line in the plane.

578. Theorem. *Two planes perpendicular to the same straight line are parallel; and conversely, if one of two parallel planes is perpendicular to a straight line, the other is also.*



Given planes P and $Q \perp AB$.

To prove $P \parallel Q$.

Outline of proof. If the planes were not parallel, they would meet. Two perpendiculars could then be drawn to the line from a point in their intersection.

CONVERSELY:

Given plane P parallel to plane Q, and $P \perp AB$.

To prove $Q \perp AB$.

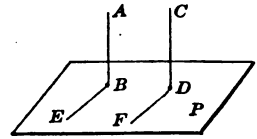
Outline of proof. AB is perpendicular to any line in P through its foot. Draw two such lines.

The planes determined by these lines and AB will intersect Q in lines parallel respectively to the lines in P, and, therefore, will be perpendicular to AB.

579. Theorem. *If two planes are parallel to a third plane, they are parallel to each other.*

Proof similar to that of § 124.

580. Theorem. *If one of two parallel lines is perpendicular to a plane, the other is also; and conversely, two straight lines perpendicular to the same plane are parallel.*



Given $AB \parallel CD$, and AB perpendicular to plane P.

To prove $CD \perp P$.

Proof. Through D draw any line DF in plane P.

Through B draw BE in plane P and parallel to DF.

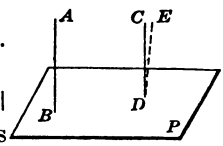
Then $\angle ABE = \angle CDF$. § 567

But $\angle ABE$ is a rt.$\angle$. Why?

Therefore $\angle CDF$ is a rt.$\angle$ and $CD \perp DF$, any line in plane P through D.

$\therefore CD \perp P$. Why?

CONVERSELY:



Given $AB \perp$ plane P, and $CD \perp$ plane P.

To prove $AB \parallel CD$.

Outline of proof. Through D draw $ED \parallel AB$. Then $ED \perp P$, and therefore coincides with CD by § 575.

581. Theorem. *Two parallel planes are everywhere the same distance apart.*

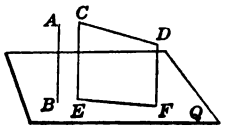
Choose any two points in one plane and draw perpendiculars to the other plane, § 574. Then these lines are perpendicular to the first plane, § 578. Further these two lines are parallel, § 580, and determine a plane intersecting the two parallel planes in parallel lines, § 562. Then apply § 191.

EXERCISES

1. Prove the theorem of § 553 by § 580.
2. A line cannot be perpendicular to each of two intersecting planes.

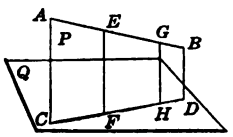
PROJECTIONS

582. Definitions. The **projection of a point** upon a plane is the foot of the perpendicular from the point to the plane. The **projection of a line** upon a plane is the locus of the projections of all points of the line upon the plane.



Thus B is the projection of point A upon plane Q, and EF is the projection of line CD.

583. Theorem. *The projection upon a plane of a straight line that is not perpendicular to the plane is a straight line.*



Given AB, a straight line not $\perp$ plane Q.

To prove that the projection of AB upon Q is a straight line.

Proof. From any point in AB, as E, draw $EF \perp Q$. § 574

Plane P, determined by AB and EF, intersects Q in CD. § 551

From G, any other point in AB, draw $GH \perp Q$. § 574

Then $GH \parallel EF$ and lies in P. Why?

Therefore its foot H lies in CD. Why?

And the projections of all points of AB lie in CD.

Furthermore, a perpendicular to Q at any point in CD will intersect AB. Why?

Then every point in CD is the projection of a point in AB.

$\therefore$ the projection of AB upon Q is a straight line.

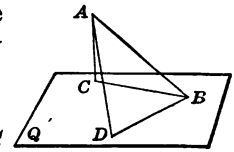
584. Theorem. *Of all oblique lines drawn from a point to a plane:*

(1) *Those that have equal projections are equal.*

(2) *Those that have unequal projections are unequal, and the one having the greater projection is the greater.*

Another statement of this theorem is: Of all oblique lines, drawn to a plane from a point in a perpendicular to the plane, those that cut off equal distances from the foot of the perpendicular are equal, and of those that cut off unequal distances from the foot of the perpendicular, the more remote is the greater.

585. Theorem. *The acute angle that a straight line makes with its projection upon a plane is the least angle that it makes with any line passing through its foot and lying in the plane.*



Given the straight line AB meeting plane Q at B, its projection CB in Q, and DB, any other line through B and in Q.

To prove $\angle ABC < \angle ABD$.

Proof. Make $BD = BC$, and draw AC and AD.

Then $\angle ABC < \angle ABD$. § 259

586. Definition. The acute angle that a straight line makes with its own projection upon a plane is called the **inclination** of the line to the plane, or the **angle that the line makes with the plane.**

EXERCISES

1. State and prove the converse theorems to § 584.

2. How does the length of the projection of a line upon a plane compare with the length of the line when: (1) the line is parallel to the plane; (2) the line is perpendicular to the plane; (3) the line is neither parallel nor perpendicular to the plane?

3. Which is the greatest angle that an oblique line to a plane makes with any line in the plane and through its foot?

4. Parallel lines make equal angles with a plane.

5. If a straight line intersects two parallel planes it makes equal angles with the parallel planes.

6. Find the projection of a line 16 in. long upon a plane if the angle it makes with the plane is 45°. If 30°. If 60°.

7. What is the locus of a point in a given plane and equidistant from two given points not in the given plane?

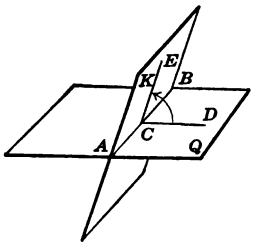
8. A rectangle 8 in. by 12 in. is intersected along one of its diagonals by a plane. If the other diagonal makes an angle of 45° with the plane, find the distance from the extremities of this diagonal to the plane.

Ans. 5.098 in.

9. The projection of a given line on each of two planes is a in length. Are the two planes necessarily parallel? Illustrate.

DIEDRAL ANGLES

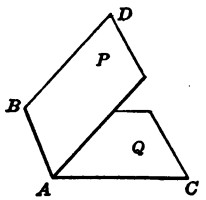
587. Definitions. An angle formed by two intersecting planes is called a **diedral angle.** The planes are called the **faces** of the diedral angle, and their intersection is called the **edge.**



If a plane Q revolves about an axis AB, and if a line $CD \perp AB$ is taken in Q, then the plane Q generates a diedral angle, and the line CD generates a plane angle since it revolves in a plane. When the plane angle is acute, right, obtuse, etc., the diedral angle generated in connection with it is acute, right, obtuse, etc. That is, the diedral angle and the plane angle are each formed by the same amount of turning.

588. Plane Angle. The plane angle generated by CD is called the **plane angle** of the diedral angle. It is formed by two lines, one in each face of the diedral angle, and perpendicular to the edge at the same point.

A diedral angle is read by naming its faces, as the diedral angle *QP;* or it may be read *C-AB-D;* or simply *AB.*



The words *complementary, supplementary, adjacent, vertical,* etc., when applied to diedral angles, have meanings corresponding to their meanings when applied to plane angles.

589. The following facts are readily deduced from the definitions:

(1) *All plane angles of a diedral angle are equal.*

(2) *If two diedral angles are equal, their plane angles are equal; and conversely.*

(3) *Two diedral angles are proportional to their plane angles; that is, the plane angle of a diedral angle can be taken as the measure of the diedral angle.*

EXERCISES

1. By comparison with the corresponding terms in plane geometry, form definitions applying to diedral angles for right, oblique, vertical, adjacent, supplementary, alternate interior. How many of these can be illustrated by the leaves of an open book?

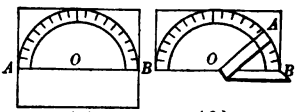
2. With reference to planes, what corresponds to the following axiom in plane geometry: Through a given point only one straight line can be drawn parallel to another straight line?

3. If one plane intersects another plane, the vertical diedral angles are equal.

4. If the sum of two adjacent diedral angles is two right diedral angles, the exterior faces lie in the same plane.

5. If two planes are cut by a third plane so as to make the alternate diedral angles equal, the two planes are parallel.

6. Take a piece of heavy paper, rule it as shown in figure (1), and cut along AO. Fold as shown in (2) and show how it may be used in measuring diedral angles.

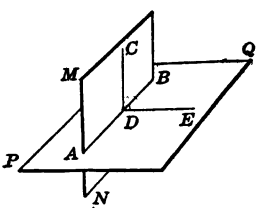


590. Perpendicular Planes. If two planes form a right diedral angle, the two planes are said to be **perpendicular** to each other.

591. Theorem. *If a line is perpendicular to a plane, every plane containing this line is perpendicular to the given plane.*

Given CD perpendicular to plane PQ, and plane MN through CD intersecting PQ in AB.

To prove that plane MN is perpendicular to plane PQ.



Proof. $CD \perp AB$. Why?

In plane PQ through D, draw $DE \perp AB$.

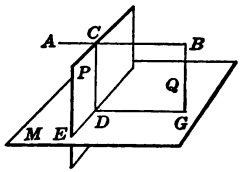
Then $\angle CDE$ is the plane angle of the diedral $\angle M\text{-}AB\text{-}Q$. § 588

But $\angle CDE$ is a rt.$\angle$. Why?

Hence diedral $\angle M\text{-}AB\text{-}Q$ is a rt. diedral $\angle$. § 589 (3)

$\therefore$ plane MN is perpendicular to plane PQ. Why?

592. Theorem. *If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.*



Given $AB \parallel$ plane M, and plane $P \perp AB$ at C.

To prove $P \perp M$.

Proof. From any point B in AB draw $BG \perp M$. § 574

AB and BG determine a plane Q which intersects plane M in DG, and plane P in CD.

Further	$DG \parallel AB$, and $CD \perp CB$.	Why?
Then	$CD \parallel BG$.	Why?
Therefore	$CD \perp M$.	§ 580
	$\therefore P \perp M$.	§ 591

EXERCISES

1. Through a given point pass a plane perpendicular to a given plane. How many such planes can be drawn?

2. If one line is perpendicular to another line, is every plane containing the first line perpendicular to the second? Prove.

3. A plane determined by a line and its projection upon a given plane is perpendicular to the given plane.

4. If three concurrent lines are each perpendicular to the other two, what can be proved concerning the planes determined by the lines? Illustrate by the walls of a room.

5. If a plane is perpendicular to a line in another plane it is perpendicular to that other plane.

6. State the converse of the theorem of § 592. Is it a true theorem? Prove.

7. A circle is divided into eight equal parts by diameters. A plane containing one of these diameters makes an angle of 45° with the plane of the circle. Find the distance of each division point of the circle from the plane.

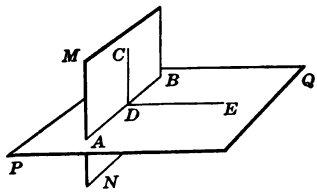
593. Theorem. *If two planes are perpendicular to each other a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*

Given plane $MN \perp$ plane PQ, AB their intersection, and $CD \perp AB$ in plane MN.

To prove $CD \perp$ plane PQ.

Proof. Through D and in plane PQ draw $DE \perp AB$.

Then $\angle CDE$ is the plane angle of the right diedral $\angle M\text{-}AB\text{-}Q$.



Hence $\angle CDE$ is a rt.$\angle$, and	$CD \perp DE$.	Why?
But	$CD \perp AB$.	Given
	$\therefore CD \perp$ plane PQ.	§ 571

594. Theorem. *If two planes are perpendicular to each other a perpendicular to one of them at any point of their intersection will lie in the other.*

Given plane $MN \perp$ plane PQ, and intersecting it in AB, also $CD \perp$ plane PQ at any point D in AB.

To prove that CD must lie in plane MN.

Suggestion. In plane MN draw a line $\perp AB$ at D. Then this line is $\perp$ plane PQ, and must coincide with CD. Why?

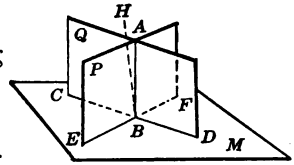
595. Theorem. *If two planes are perpendicular to each other a perpendicular to one from any point in the other will lie in the other.*

Given plane $MN \perp$ plane PQ, and intersecting it in AB, also $CD \perp$ plane PQ from any point C in plane MN.

To prove that CD must lie in plane MN.

Suggestion. In plane MN draw a line $\perp AB$ from C. Then this line is $\perp$ plane PQ, and must coincide with CD. Why?

596. Theorem. *If each of two intersecting planes is perpendicular to a third plane their line of intersection is perpendicular to the third plane.*



Given planes P and Q, intersecting in AB, and each $\perp$ plane M.

To prove $AB \perp M$.

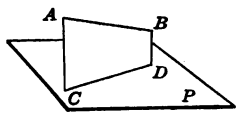
Proof. Either $AB \perp M$ or it is not.

If AB is not $\perp M$, draw a line $HB \perp M$ at B.

Then HB lies in both P and Q.	§ 594
Hence HB is the intersection of P and Q.	Why?
But AB is the intersection of P and Q.	Given
Therefore AB and HB coincide.	§ 551

$\therefore AB \perp M$.

597. Theorem. *Through any straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to the given plane.*



Given plane P and AB a line not $\perp P$.

To prove that one plane $\perp P$ can be passed through AB, and only one.

Proof. From any point A in AB, draw $AC \perp P$.

AB and AC determine a plane $\perp P$.	§ 591

$\therefore$ one plane $\perp P$ can be passed through AB.

If another plane $\perp P$ could be drawn through AB, then AB would be $\perp P$ by § 596.

But AB is not $\perp P$.	Given

$\therefore$ only one plane $\perp P$ can be passed through AB.

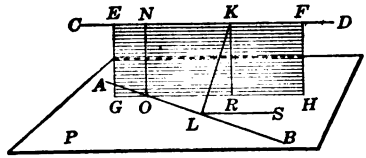
EXERCISES

1. Discuss the theorem of § 597 for the case where AB lies in plane P.

2. If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.

3. Any point in a plane containing the bisector of an angle, and perpendicular to the plane of the angle, is equally distant from the sides of the angle.

598. Theorem. *Between two straight lines not in the same plane one common perpendicular, and only one, can be drawn.*



Given AB and CD two non-coplanar straight lines.

To prove that one common perpendicular to AB and CD, and only one, can be drawn.

Proof. Through AB pass plane $P \parallel CD$. § 559

Through CD pass plane $EH \perp P$, and intersecting P in GH. § 597

Then $GH \parallel CD$. Why?

And GH intersects AB in some point O. Why?

In plane EH draw $NO \perp GH$ at O.

Then $NO \perp CD$ and $NO \perp AB$. Why?

$\therefore$ one common perpendicular to AB and CD can be drawn.

Suppose that there is another common perpendicular KL.

In plane P draw $LS \parallel GH$, and in plane EH draw $KR \perp GH$.

Then $LS \parallel CD$, and hence $KL \perp LS$. Why?

Therefore $KL \perp$ plane P. Why?

But this is impossible for $KR \perp$ plane P. Why?

Hence KL is not $\perp AB$.

$\therefore$ only one common perpendicular to AB and CD can be drawn.

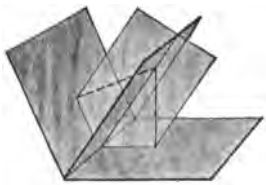
599. Theorem. *The common perpendicular between two non-coplanar lines is the shortest line that can be drawn between them.*

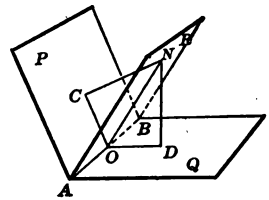
Prove that NO is less than any other line KL, for $NO = KR$ and $KR < KL$.

600. Definition. The length of the shortest line between two lines is called the **distance** between them.

Compare with §§ 187-190, 577.

601. Theorem. ***Every point in a plane that bisects a diedral angle is equidistant from the faces of the angle.***





Given the plane R bisecting the diedral angle formed by the planes P and Q, and N any point in plane R. Also $NC \perp P$ and $ND \perp Q$.

To prove $NC = ND$.

Proof. NC and ND determine plane CD. Why?

Plane CD intersects P in CO, Q in DO, and R in NO.

Plane $CD \perp P$ and also $\perp Q$. § 591

Then plane $CD \perp AB$. Why?

And NO, CO, and DO are each $\perp AB$. Why?

Therefore $\angle NOC$ is the plane angle of diedral $\angle RP$, and $\angle NOD$ is the plane angle of diedral $\angle RQ$.

Hence $\angle NOC = \angle NOD$. Why?

Show $\triangle NOC \cong \triangle NOD$.

$\therefore NC = ND$. Why?

602. Theorem. *Every point equidistant from the two faces of a diedral angle lies in the plane bisecting the angle.*

EXERCISES

1. Are both of the preceding theorems included in the following? The locus of a point equidistant from the faces of a diedral angle is the plane bisecting the angle. Explain.

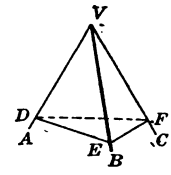
2. From any point within a diedral angle perpendiculars are drawn to the faces. Prove that the angle formed by these perpendiculars is supplementary to the plane angle of the diedral angle.

3. The plane angle of a diedral angle is 120°. A point in the bisector of the diedral angle is 16 in. from the edge of the angle. Find the distance of this point from the faces of the angle.

POLYEDRAL ANGLES

603. Definitions. When three or more planes meet at a point they form a **polyedral angle,** or a **solid angle.**

The polyedral angle is formed by a portion of the planes as shown in the figure. The point V in which all the planes meet is the **vertex** of the polyedral angle; the lines, AV, BV, and CV, of intersection of consecutive planes are the **edges;** the portions of the planes lying between the edges are the **faces;** and the angles in the faces between the edges are the **face angles.**



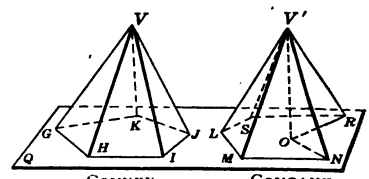
A polyedral angle is read by the letter at the vertex, or by this letter together with a letter on each edge.

Thus, the polyedral angle in the figure is read "polyedral angle V, or V-ABC."

604. Parts of a Polyedral Angle. The face angles are AVB, BVC, and CVA. The diedral angles formed by the faces are the diedral angles of the polyedral angle. The **parts** of a polyedral angle are its face angles and its diedral angles.

For convenience in representing a polyedral angle, a plane is often passed through it cutting all its edges. It should be noted that this plane is not a part of the polyedral angle.

605. Convex and Concave Polyedral Angles. If a plane cuts all the edges of a polyedral angle, but not through the vertex, the intersections of the faces with this plane form a polygon. If this polygon is convex (§ 169), the polyedral angle is said to be **convex.** If the polygon is concave, the polyedral angle is said to be **concave.**

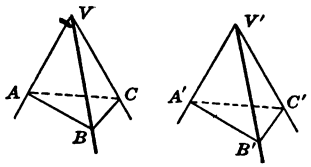


In this text only convex polyedral angles will be considered, unless otherwise stated.

606. Classification. A polyedral angle that has three faces is called a **triedral angle.** The words **tetraedral, pentaedral, hexaedral,** etc., may be applied when the polyedral angle has four, five, six, etc., faces.

607. Congruent Polyedral Angles. If the corresponding parts of two polyedral angles are equal and arranged in the same order, the polyedral angles are **congruent.**

Thus, in the figure, the face angles are equal and arranged in the same order, that is, $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, and $\angle CVA = \angle C'V'A'$. Also in the diedral angles, $\angle AV = \angle A'V'$, $\angle BV = \angle B'V'$, and $\angle CV = \angle C'V'$, the arrangement is in the same order.



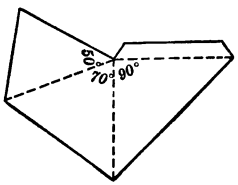
If the corresponding parts of two polyedral angles are equal and arranged in the opposite order, the polyedral angles are said to be **symmetric.** These will be considered later (§ 836).

EXERCISES

1. Does the size of a polyedral angle depend upon the lengths of its edges? How do the number of edges, the number of diedral angles, and the number of face angles of a polyedral angle compare?

2. Construct from stiff paper a triedral angle having face angles equal to 50°, 70°, and 90°. Can you tell the number of degrees in the diedral angles?

The paper may be cut as indicated in the figure, folded along the dotted lines and pasted.



3. How many degrees in each of the face angles and the diedral angles of the polyedral angle formed by the walls of a room?

4. The face angles and diedral angles of a convex polyedral angle are each less than 180°.

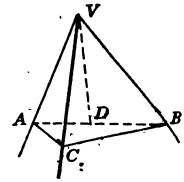
5. Bearing the plane geometry definitions in mind, define vertical polyedral angles. Are they congruent or symmetric? Explain how, if it is possible to have two vertical triedral angles congruent.

6. Could a triedral angle have one right diedral angle? Could it have two? Three? Give illustrations.

608. Theorem. ***The sum of two face angles of a triedral angle is greater than the third face angle.***

Given the triedral $\angle V\text{-}ABC$.

To prove that the sum of two face angles is greater than the third face angle.



Proof. Three cases arise: (1) all the face angles equal; (2) two face angles equal; (3) no two face angles equal.

Cases (1) and (2) are left to the student.

In case (3) suppose $\angle AVB$ is the greatest.

In the face AVB, construct $\angle AVD = \angle AVC$, take $VD = VC$, and draw AB, AC, and BC.

Then	$\triangle AVC \cong \triangle AVD$.	§ 247
And	$AC = AD$.	Why?
	$AC + CB > AD + DB$.	§ 182
Hence	$CB > DB$.	§ 174

In $\triangle CVB$ and DVB, $BV = BV$, and $CV = DV$, but $CB > DB$.

Therefore	$\angle CVB > \angle DVB$.	§ 259
Then	$\angle AVC + \angle CVB > \angle AVD + \angle DVB$.	§ 173
	$\therefore \angle AVC + \angle CVB > \angle AVB$.	

Or the sum of two face angles is greater than the third.

EXERCISES

1. Any face angle of a triedral angle is greater than the difference of the other two.

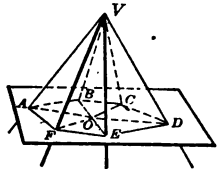
2. State the theorems in plane geometry that correspond to the theorems of § 608 and Exercise 1.

There is a close analogy between the plane triangle and the triedral angle. Many theorems concerning plane triangles can be changed to theorems concerning triedral angles by replacing the word *side* by *face angle*, and the word *angle* by *diedral angle*.

3. State theorems for triedral angles analogous to the theorems of plane geometry that have to do with congruent triangles.

4. Referring to the figure of § 608, $\angle AVC = 55°$ and $\angle CVB = 65°$. Make a statement regarding the number of degrees in $\angle AVB$.

609. Theorem. ***The sum of the face angles of any convex polyedral angle is less than four right angles.***



Given the convex polyedral $\angle V\text{-}ABCDEF$.

To prove $\angle AVB+\angle BVC+\cdots\angle FVA<4$ rt.$\measuredangle$.

Proof. Let a plane intersect the edges of the polyedral angle in A, B, C, etc., and intersect the faces in AB, BC, CD, etc. Join any point in the polygon thus formed to the vertices of the polygon.

$\therefore$ Then $\angle VBA+\angle VBC>\angle ABC$. § 608

Similarly $\angle VCB+\angle VCD>\angle BCD$, etc.

Add these and the result is that the sum of all the base angles of the triangles with vertices at V is greater than the sum of all the base angles of the triangles with vertices at O.

Then, since the sum of all the angles of the triangles with vertices at V is equal to the sum of those with vertices at O, the sum of the angles about V is less than the sum of the angles about O. § 178

But the sum of the angles about $O=4$ rt.$\measuredangle$. Why?

$\therefore \angle AVB+\angle BVC+\cdots\angle FVA<4$ rt.$\measuredangle$.

EXERCISES

1. Can a polyedral angle have for its face angles three angles from an equilateral triangle? Can it have four? Five? Six? Why?

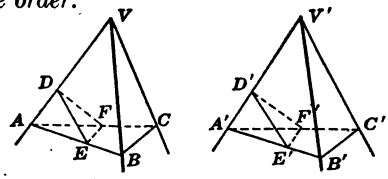
2. Can a polyedral angle have as face angles the angles of a square? Of a regular pentagon? Of a regular hexagon? Why?

3. Any face angle of a polyedral angle is less than the sum of the others.

4. The sum of the face angles of a polyedral angle is 300°. What is the greatest value that any one of the face angles may have?

5. One of the face angles of a triedral angle is 60°. What are the limitations on the size of each of the other face angles?

610. Theorem. *Two triedral angles are congruent if the three face angles of one are equal respectively to the three face angles of the other, and arranged in the same order.*



Given triedral angles V and V', having $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$, and arranged in the same order.

To prove triedral $\angle V \cong$ triedral $\angle V'$.

Proof. On the edges of triedral angles V and V' lay off the six equal segments VA, VB, VC, $V'A'$, $V'B'$, and $V'C'$, and draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then there are three pairs of congruent isosceles triangles as follows: $\triangle AVB \cong \triangle A'V'B'$, $\triangle BVC \cong \triangle B'V'C'$, $\triangle CVA \cong \triangle C'V'A'$. Why?

Hence $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. Why?

And $\triangle ABC \cong \triangle A'B'C'$. Why?

Also $\measuredangle VAC$ and VAB are acute. Why?

From any point D in VA construct DE in face AVB and DF in face AVC, each $\perp VA$. These lines meet AB and AC respectively in E and F. Why?

Draw EF.

Take $V'D' = VD$ and in a similar manner construct, $D'E'$, $D'F'$, and $E'F'$.

Then, since $AD = A'D'$ and $\angle DAE = \angle D'A'E'$,

$\triangle ADE \cong \triangle A'D'E'$ § 97

Hence $AE = A'E'$ and $DE = D'E'$. Why?

Similarly $AF = A'F'$ and $DF = D'F'$.

Therefore $\triangle AEF \cong \triangle A'E'F'$. § 247

And $EF = E'F'$. Why?

Therefore $\triangle DEF \cong \triangle D'E'F'$. § 113

And $\angle EDF = \angle E'D'F'$. Why?

Hence diedral $\angle VA$ = diedral $\angle V'A'$. § 589 (3)

Similarly, prove the other two pairs of diedral angles equal.

$\therefore$ triedral $\angle V \cong$ triedral $\angle V'$. § 607

LOCI

611. As has been assumed on the previous pages, the idea of a locus here is the same as in plane geometry.

Here, too, the proof of a locus theorem must, as in plane geometry, consist of two parts:

(1) *That all points in the figure satisfy the given conditions.*

(2) *That all points that satisfy the given conditions are in the figure.*

In the place of (2) one may prove that all points not in the figure do not satisfy the given conditions.

612. In plane geometry it was noticed that usually a locus satisfying one condition is a line or group of lines, and a locus satisfying two conditions is a point or group of points. There, however, a further condition is assumed, namely, that the locus is confined to a plane.

In solid geometry usually, one condition will confine the locus to a surface or group of surfaces, two conditions will confine the locus to a line or group of lines, and three conditions will confine the locus to a point or group of points.

Here, as in plane geometry, one of the greatest benefits derived from the study of loci is through the imaging and constructing figures, rather than through the proofs of the loci theorems. This, however, does not mean that the proofs should be neglected.

EXERCISES

1. What is the locus of points equally distant from two given points?

2. What is the locus of points equally distant from three given points?

3. What is the locus of points equally distant from four given points?

4. What is the locus of points equally distant from two intersecting lines and in a given plane?

5. What is the locus of the end point of a line-segment of fixed length, that moves so as to remain parallel to a given line and have one end in a given plane?

6. What is the locus of points in a given plane equally distant from three given points not in a line?

7. What is the locus of points equally distant from two given parallel lines?

8. What is the locus of the middle point of a line-segment of fixed length, that moves so as to have its end points in two parallel planes?

9. What is the locus of a point equally distant from two parallel planes and equally distant from the faces of a diedral angle?

10. What is the locus of all lines that pass through a given point and are parallel to a given plane not containing the point?

11. What is the locus of the points within a diedral angle and equally distant from its two faces?

12. What is the locus of all points that are twice as far from a line in a plane as from the plane?

13. What is the locus of the points within a triedral angle and equally distant from its three faces?

14. What is the locus of the points equally distant from the edges of a triedral angle?

15. What is the locus of a line making equal angles with each of two intersecting lines? What other exercise in this list has the same locus as this?

16. What is the locus of points in one of two given non-coplanar lines equally distant from two given points in the other?

17. What is the locus of a point that moves so that its distance from one of two given parallel planes is to its distance from the other as 1 : 3?

18. What is the locus of a point whose distances from the faces of a diedral angle are respectively 5 in. and 8 in.?

19. What is the locus of the projections of a given point upon the planes containing a given line?

Many questions concerning loci cannot be discussed in elementary geometry as they lead to forms not there considered, for in elementary geometry only a limited number of forms such as lines, circles, planes, cones, cylinders, and spheres are considered. The question: What is the locus of the middle points of all transversals of two non-coplanar lines? leads to none of these.

In general, the locus of points equally distant from two different elements, as a point and a line, a line and a plane, or two non-coplanar lines cannot be considered in elementary geometry.

QUESTIONS

1. State the different relations that two lines may have to each other. That a line may have to a plane. That two planes may have to each other.

2. State the different ways of determining that a straight line lies in a plane.

3. What is the intersection of a line and a plane? Of two planes? Of three planes?

4. State the different ways of determining that a line is parallel to a plane.

5. State the different ways of determining that a line is perpendicular to a plane.

6. State the different ways of determining that two planes are parallel.

7. State the different ways of determining that two planes are perpendicular.

8. How many intersecting lines can be parallel to the same line? To the same plane?

9. How many intersecting planes can be parallel to the same line? To the same plane?

10. Must planes be parallel if they contain parallel lines? Are lines necessarily parallel if they are in parallel planes?

11. How many lines through a point can be perpendicular to the same line? To the same plane?

12. How many intersecting planes can be perpendicular to the same line? To the same plane?

13. What are the relations of the following to each other: (1) lines parallel to the same line; (2) planes parallel to the same plane; (3) lines parallel to the same plane; (4) planes parallel to the same line; (5) lines perpendicular to the same line; (6) lines perpendicular to the same plane; (7) planes perpendicular to the same line; (8) planes perpendicular to the same plane?

14. If two planes are perpendicular to each other, what lines in one are perpendicular to the other?

15. Can the projection of a straight line upon a plane be a curve? Can the projection of a circle be a straight line? Explain.

16. Can the projection of a square be a square? A rectangle? A parallelogram? Explain.

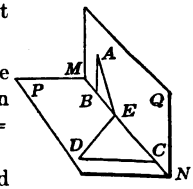
GENERAL EXERCISES

COMPUTATION

1. Two fixed points, A and B, are 10 in. apart. A point P moves so as to keep 8 in. from A and B. Find the length of the locus of P.

2. Two fixed points, A and B, are 12 in. apart. A point P moves so as to keep 10 in. from A and 4 in. from B. Find the radius of the circle generated. *Ans.* 3.13+ in.

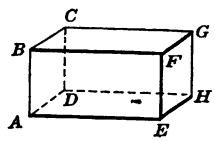
3. A 20-foot pole is placed obliquely in a river. One end of the pole is on the bottom of the river and the other end is 3 ft. above the surface. Find the depth of the river if the length of the part of the pole under water is 15 ft.



4. Plane Q is perpendicular to plane P. Find the shortest line AED that can be drawn from a point A in Q to a point D in P if $AB=5$ ft., $CD=10$ ft., and $BC=20$ ft.

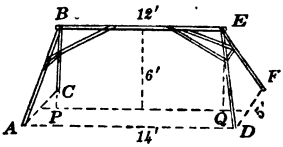
SUGGESTION. Consider the figure when Q is turned into the same plane as P.

5. The room shown in the figure has a length of 40 ft., a width of 20 ft., and a height of 12 ft. Find the length of the shortest line in the walls of the room from B to H.

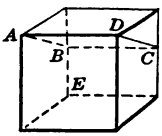


6. The sum of the face angles of a triedral angle is 148°. What is the greatest value a face angle can have?

7. In the figure of a hammock support with the dimensions as given, find the length of AB if $AB=CB=DE=FE$.

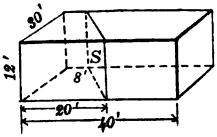


8. The roof of a house makes an angle of 30° with the plane of the plates. The roof is 18 ft. by 30 ft. Find the area of its projection upon the plane of the plates.

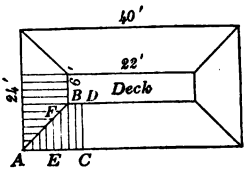


9. In the figure of a cube an edge is 18 in. and BE is 10 in. Find the area of $ABCD$.

10. Having given the dimensions as shown in the figure, find the area of section S to two decimal places. All the face planes that meet are perpendicular to each other.



11. The figure shows the plan for a half pitch roof. The dimensions are as given and the rafters are to be placed 1 ft. 6 in. from center to center. Find the lengths of the hip rafters, common rafters, and jack rafters. AB is a hip rafter, CD is a common rafter, and EF is a jack rafter.



THEOREMS AND PROBLEMS

1. If a series of parallel planes intercept equal segments on one straight line, they intercept equal segments on any other straight line that they intersect.

2. If a straight line and a plane are each perpendicular to the same straight line, they are parallel.

3. If two planes are perpendicular to each other, a line perpendicular to one and not in the other is parallel to the other.

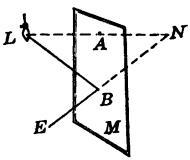
4. Prove that the sides of an isosceles triangle make equal angles with any plane in which the base lies.

5. From a point within a triedral angle perpendiculars are drawn to the three faces. These perpendiculars are the edges of a second triedral angle. Prove that the face angles of one are supplementary to the diedral angles of the other.

6. Three planes, M, N, and Q, each perpendicular to the other two pass through a common point. Show that a point 5 in. from M, 8 in. from N, and 10 in. from Q may be in any one of eight positions.

7. Given two non-coplanar lines, to construct a plane upon which the projections of the two lines will be parallel.

8. A ray of light from the source L is reflected by the mirror M and enters the eye at E. The path travelled by the light is the shortest possible path from L to the mirror and then to E. Show how to determine where the ray of light strikes the mirror.



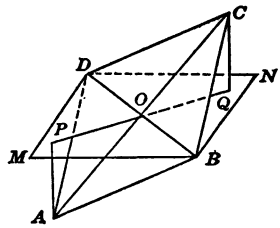
9. Three non-coplanar lines meet at a point. Construct a line that will make equal angles with the three lines.

10. Given a polyedral angle with four faces, to construct a plane that will intersect the four faces so that the intersection will be a parallelogram.

SUGGESTION. Pass a plane parallel to one face and intersecting the opposite face. Determine a line in the first face equal to this intersection and parallel to it. These two lines are parallel and hence determine a parallelogram.

11. Prove that if a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

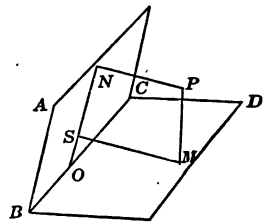
12. If a plane be passed through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.



Plane MN passes through diagonal BD of parallelogram $ABCD$. Prove perpendiculars AP and CQ are equal.

13. Describe how to locate a point that is 4 in. from face of a triedral angle, 5 in. from another face, and 3 in. from the third face.

14. If a series of parallel planes cut all the edges of a triedral angle, the intersections of the planes with the faces form similar triangles.

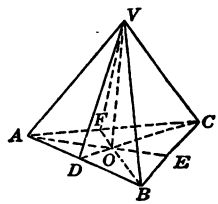


15. If from P, any point within the diedral angle A-BC-D, PM and PN are drawn perpendicular to the faces BD and AC respectively, and MS is drawn perpendicular to AC, then NS is perpendicular to BC.

16. The three planes bisecting the diedral angles of a triedral angle meet in a line.

SUGGESTION. Show that two of them meet in a line, and then show that this line lies in the third plane.

Compare this exercise with Exercise 13, page 308.



17. All points within a triedral angle and equally distant from its three faces, lie in the line of intersection of the planes that bisect the diedral angles.

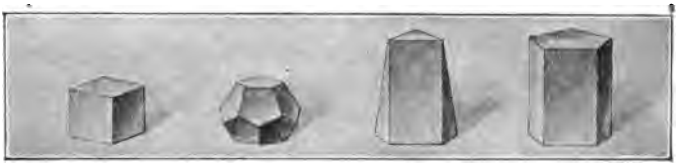
18. The planes through the bisectors of the face angle of a triedral angle and perpendicular to the planes of the respective faces, meet in a line.

CHAPTER VII

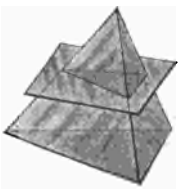
POLYEDRONS, PRISMS, CYLINDERS

613. In the present and following chapters will be considered some of the solids most commonly observed in nature, and very frequently used in architecture, engineering, and the arts. Besides finding the areas and volumes of these solids it will be necessary to investigate the relations of their parts, and to study the plane figures formed when the solids are cut by planes.

614. Polyedrons. A **polyedron** is a solid entirely bounded by planes.



The intersections of the bounding planes are the **edges** of the polyedron. The points in which the edges intersect are the **vertices.** The polygons bounded by the edges are the **faces.** The faces taken together make up the **surface,** and the area of this surface is the **area** of the polyedron. The amount of space enclosed by the surface is the **volume** of the polyedron.

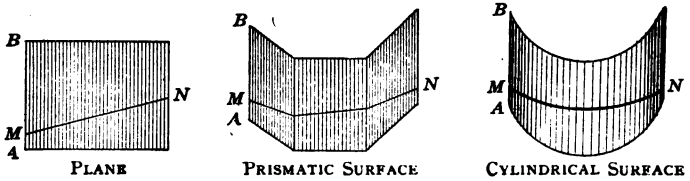


615. Sections. The plane figure formed on a plane passing through a solid, and bounded by the intersections of the plane with the surface of the solid is called a **section of the solid.** It is evident that the section of a solid is a **closed figure** (§ 146), and that the section of a polyedron is a *polygon.*

616. Convex Polyedrons. A polyedron is **convex** if every section of it is a convex polygon. Otherwise it is **concave.**

Only convex polyedrons are considered unless otherwise stated.

617. Prismatic and Cylindrical Surfaces. A moving straight line that always remains parallel to its original position, and intersects a fixed straight line, generates a **plane.** Why?



PLANE PRISMATIC SURFACE CYLINDRICAL SURFACE

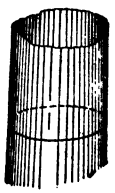
A moving straight line that always remains parallel to its original position, and intersects a broken line not coplanar with it, generates a **prismatic surface.**

A moving straight line that always remains parallel to its original position, and intersects a plane curved line not coplanar with it, generates a **cylindrical surface.**

AB is the original position of the moving line, and *MN* is the fixed line.

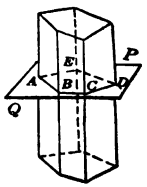
618. The moving line is called the **generatrix,** and the fixed line or curve the **directrix.**

The generatrix in any position is called an **element** of the surface generated.



619. Closed Surface. If the directrix is a closed line, the prismatic or cylindrical surface is **closed.**

620. Sections of Prismatic and Cylindrical Surfaces. A plane cutting all the elements of a closed prismatic or cylindrical surface cuts the surface in a closed line. The figure bounded by this closed line is a **section.** If the cutting plane is perpendicular to an element, the section is a **right section;** if not, it is an **oblique section.** Two sections made by parallel planes are **parallel sections.**

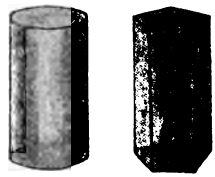


In the figure, *ABCDE* is a right section.

PRISMS AND CYLINDERS

621. A **prism** is the solid formed by a closed prismatic surface and two parallel cutting planes.

622. A **cylinder** is the solid formed by a closed cylindrical surface and two parallel cutting planes.

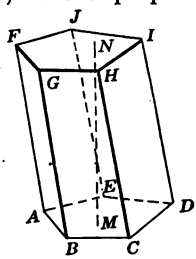


623. The sections formed by the cutting planes are called the **bases** of the prism or cylinder. It follows from § 620 that the bases of a prism are polygons.

624. The polygons formed on the prismatic surface between the bases are called the **lateral faces.** Intersections of the lateral faces are **lateral edges,** and intersections of the lateral faces with the bases are **base edges.**

625. The **altitude** of a prism, or cylinder, is the perpendicular distance between its bases.

626. The **lateral area** of a prism, or cylinder, is the prismatic, or cylindrical, surface between the bases. The **total area** is the lateral area together with the areas of the bases.



In the figure read the bases, faces, lateral edges, base edges, altitude.

627. The following facts about prisms and cylinders are readily proved by the definitions and previous theorems:

(1) *The lateral edges of a prism, or the elements of a cylinder, are equal.*

(2) *The lateral faces of a prism are parallelograms.*

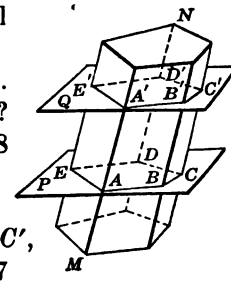
(3) *The right section of a prism, or cylinder, is perpendicular to all the lateral edges, or elements.*

(4) *The section of a prism made by a plane parallel to a lateral edge is a parallelogram.*

(5) *The section of a cylinder made by a plane containing two elements is a parallelogram.*

628. Theorem. ***Parallel sections of a prism are congruent.***

Given the prism MN, and the parallel sections AD and $A'D'$.



To prove polygon $AD \cong$ polygon $A'D'$.

Proof. $AB \parallel A'B'$. Why?

Hence $AB = A'B'$. § 158

Similarly $BC = B'C'$, $CD = C'D'$,

$DE = D'E'$, $EA = E'A'$.

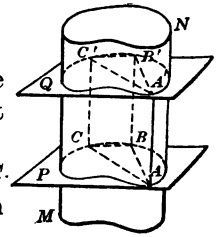
Also $\angle EAB = \angle E'A'B'$, $\angle ABC = \angle A'B'C'$, etc. § 567

The two polygons AD and $A'D'$ are then mutually equilateral and mutually equiangular. They then can be proved congruent by superposition.

$\therefore$ polygon $AD \cong$ polygon $A'D'$.

629. Theorem. *Parallel sections of a cylinder are congruent.*

Given the cylinder MN, and the parallel sections P and Q made by the planes.



To prove $P \cong Q$.

Proof. Take two points A and B on the perimeter of P, and let C be any third point of that perimeter.

Draw the elements through A, B, and C to the points A', B', and C' respectively in the perimeter of Q; and draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then $\triangle ABC \cong \triangle A'B'C'$. Why?

And, if P is superposed on Q with AB coinciding with $A'B'$, C will fall upon C'. Why?

But C is any point in the perimeter of P.

Hence every point in the perimeter of P will fall upon a point of Q, and conversely.

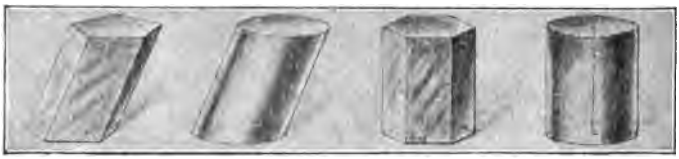
$\therefore P \cong Q$.

630. Theorem. *Every section of a prism, or cylinder, parallel to the bases is congruent to them.*

631. A **right prism,** or **cylinder,** is one whose bases are right sections.

632. An **oblique prism,** or **cylinder,** is one whose bases are oblique sections.

633. A prism is **triangular, quadrangular, pentagonal,** etc., according as its bases have *three, four, five,* etc., sides.



634. A **regular prism** is a right prism whose bases are regular polygons.

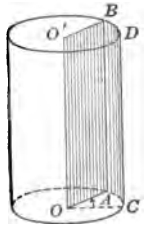
635. A **circular cylinder** is a cylinder whose bases are enclosed by circles. If the bases are also right sections, the cylinder is a **right circular cylinder.**

636. The line joining the centers of the bases of a circular cylinder is called its **axis.** The radius of the base of a circular cylinder is called the **radius** of the cylinder.

637. Theorem. *A lateral edge of a right prism, or an element of a right cylinder, is equal to the altitude.* §§ 620, 563.

638. Theorem. *The lateral faces of a right prism are rectangles.* Apply § 627 (2).

639. Theorem. *If a rectangle revolves about one of its sides as an axis, it generates a right circular cylinder; and conversely.*



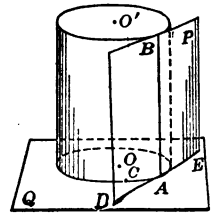
Show that the side opposite the axis remains parallel to its original position and moves along a curve. Also show that the other two sides generate parallel bases that are circular, and are perpendicular to the elements.

640. Cylinder of Revolution. Since a right circular cylinder may be generated by revolving a rectangle about one of its sides as an axis, it is often called a **cylinder of revolution.**

641. Plane Tangent to a Cylinder. A plane is **tangent** to a cylinder if it touches the cylindrical surface in one element only.

642. Theorem. *If a plane is tangent to a circular cylinder, its intersection with the plane of a base is tangent to that base.*

Given the cylinder OO' and the tangent plane P, meeting the cylinder in the element AB, and intersecting the plane of base O in DE.



To prove that DE is tangent to base O.

Proof. DE has one point in common with $\odot O$. Why?

Also DE has only one point in common with $\odot O$, for if it had another point as C in common with $\odot O$, plane P would have a point outside the element AB in common with the cylinder, which is impossible. Why?

DE also lies in the plane of $\odot O$. Why?

$\therefore$ DE is tangent to $\odot O$. § 285

643. Theorem. *The plane determined by a tangent to a base of a circular cylinder and the element through the point of contact is tangent to the cylinder.*

EXERCISES

1. The bases of prisms, or cylinders, are congruent.

2. Would the theorem of § 628 be true if the sections were concave?

3. A line drawn parallel to the elements of a circular cylinder from the center of one base, intersects the other base at its center.

4. The axis of a circular cylinder is parallel to its elements.

5. The axis of a circular cylinder passes through the center of all sections parallel to the bases.

6. Every lateral edge of a prism is parallel to the plane determined by any other two lateral edges.

7. What is the locus of all points equidistant from a given straight line?

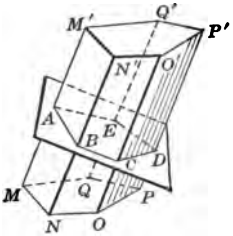
8. Show that the plane determined by an element of a circular cylinder and the center of one base will contain the axis.

AREAS OF PRISMS AND CYLINDERS

644. Theorem. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*

Given the prism MP', and its right section $ABCDE$.

To prove $S=pe$, where S denotes lateral area, p the perimeter of a right section, and e a lateral edge.



Proof. Area of $\square MN'=e\cdot AB$. § 355

Similarly area of $\square NO'=e\cdot BC$,

etc.

Since the lateral area is the sum of the faces, adding these gives:

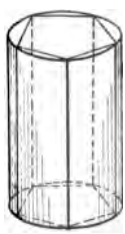
$$S=e(AB+BC+\cdots EA). \qquad \text{Why?}$$

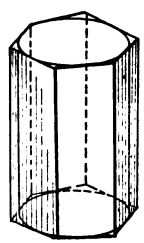
But $$AB+BC+\cdots EA=p. \qquad \text{Why?}$$

$$\therefore S=pe. \qquad \text{Why?}$$

645. Theorem. ***The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.***

That is, $S=ph$, where S denotes lateral area, p perimeter of base, and h altitude.





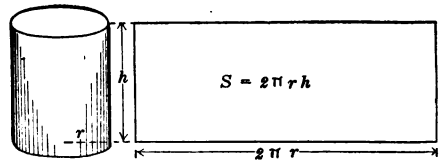
646. Definitions. An **inscribed prism** is a prism whose bases are respectively inscribed in the bases of a cylinder. The cylinder is then said to be **circumscribed** about the prism.

A **circumscribed prism** is a prism whose bases are respectively circumscribed about the bases of a cylinder. The cylinder is then said to be **inscribed** in the prism.

647. It is evident that the lateral edges of an inscribed prism are elements of the cylinder; and that the lateral faces of the circumscribed prism are tangent to the cylinder by § 641.

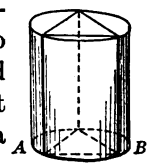
648. Area of Cylinder. As in § 480, with the circle, the general idea of the measurement of a plane area cannot be used here, for it is impossible to apply a plane unit of area to a cylindrical surface and thus find its numerical measure.

Practically, however, the lateral area of a right circular cylinder can be found by considering the surface as straightened out and measured as a rectangle having a length equal to the circumference of the base, and with a width equal to the altitude of the cylinder.



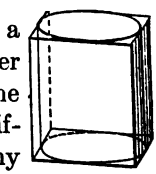
It remains to devise a method for measuring the surface of a cylinder, which is accurate and also geometric.

649. In the circular cylinder of base AB, a prism having a regular polygon for base is inscribed, and the number of sides is doubled indefinitely according to § 463. The area of the cylinder is a constant and that of the prism is a variable. It is evident that the area of each prism is greater than the area of the prism of half the number of faces.



It is also evident that, at all times, the area of the cylinder is greater than the area of each successive prism. By doubling the number of faces of the prism indefinitely, the area continually increases, and can be made to differ from the area of the cylinder by less than any assigned value, that is, *it can be made just as nearly equal to the area of a cylinder as desired.*

Similarly, by circumscribing a prism having a regular polygon for base about a circular cylinder and doubling the number of faces indefinitely, the area continually decreases, and can be made to differ from the area of the cylinder by less than any assigned value, that is, *it can be made just as nearly equal to the area of the cylinder as desired.*



650. Volume. Further, it is evident that the method of the preceding paragraphs can be as readily applied to the volume of a cylinder as to its area. It is thus seen that the volume of an inscribed or circumscribed prism, having a regular polygon for base, as the number of faces is indefinitely doubled, *can be made just as nearly equal to the volume of the cylinder as desired.*

651. From the preceding discussion, the following theorem may be considered established:

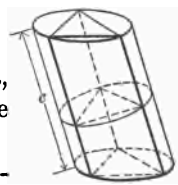
Theorem. *If prisms whose bases are regular polygons are inscribed in and circumscribed about a circular cylinder, and if the number of faces of the prisms is indefinitely doubled,*

(1) *The perimeter of a right section of the cylinder is the common limit of the perimeters of right sections of the prisms.*

(2) *The lateral area of the cylinder is the common limit of the lateral areas of the prisms.*

(3) *The volume of the cylinder is the common limit of the volumes of the prisms.*

652. Theorem. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section and the length of an element.*



Given a circular cylinder.

To prove $S = pe$, where S denotes lateral area, p the perimeter of a right section, and e the length of an element.

Proof. Inscribe a prism, whose base is a regular polygon, in the cylinder, and let S' denote its lateral area and p' the perimeter of a right section.

By doubling indefinitely the number of faces of the prism,
$S' \to S$, and $p' \to p$. § 652 (1) and (2)

Then $p'e \to pe$. § 485 (2)

But $S' = p'e$, being variables that are always equal. § 644

$\therefore S = pe$. § 485 (1)

653. Theorem. ***The lateral area of a right circular cylinder is equal to the product of its altitude and the circumference of its base.***

That is, $S = ch$, where S denotes lateral area, c circumference of base, and h altitude.

EXERCISES

1. Show that the lateral area and the total area of a right circular cylinder of altitude h and radius r, are given by the formulas:
$S = 2\pi rh$, $T = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$.

In the following use the notation of Exercise 1.

2. Given $r = 8$ and $h = 10$; find S and T.

3. Given $r = 10$ and $S = 200$; find h and T.

4. Given $h = 16$ and $S = 256$; find r and T.

5. Given $r = 12$ and $T = 4800$; find h and S.

6. Given $h = 25$ and $T = 4500$; find r and S.

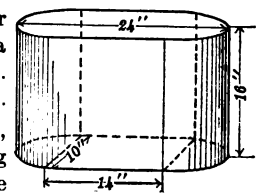
7. Given $S = 2000$ and $T = 2200$; find r and h.

8. Solve the formula $T = 2\pi rh + 2\pi r^2$, (1) for r in terms of h and T, (2) for h in terms of r and T.

9. From the formulas of Exercise 1, find r in terms of S and T.

10. A peck measure made of sheet iron has a diameter of 8 in. and a depth of 10.7 in. Find the number of square inches of sheet iron in it.

11. Find the area (no cover) of a wash boiler if the bottom is in the form of a rectangle with a semicircle at each end. The rectangle is 10 in. by 14 in., and the depth of the boiler is 16 in.



12. A steam boiler has a diameter of 72 in., is 18 ft. long. and contains 70 tubes each having a diameter of 4 in. extending lengthwise of the boiler. Find the heating surface of the boiler, using "one-half" in the rule below. *Ans.* 1505 sq. ft.

RULE. In finding the heating surface of a horizontal boiler, it is customary to take one-half to two-thirds the lateral area of the shell, the lateral area of the tubes, one-half to two-thirds the area of the ends of the boiler, and subtract the area of both ends of the tubes.

13. Find the heating surface of a horizontal tubular boiler 12 ft. long, 5 ft. in diameter, and having 52 tubes $2\frac{1}{2}$ in. in diameter. Use "two-thirds" in the preceding rule. *Ans.* 556.7 sq. ft.

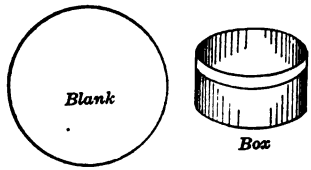
14. The base of a right prism is a triangle whose sides are 12 ft., 15 ft., and 17 ft., and its altitude is $8\frac{1}{2}$ ft. Find its lateral area and total area.

15. Find the lateral area of a regular hexagonal prism each side of whose base is 4 in. and whose altitude is 16 in. Find the total area.

16. Derive the formula $S = 0.2618dl$, where S is the area of the surface of a cylindrical pipe in square feet, d the diameter in inches, and l the length in feet.

17. Six lines of steam pipes of diameter 2 in. extend along one side and an end of a room 40 ft. by 30 ft. Find the number of square feet of heating surface.

18. Small metal boxes used for various purposes are made from sheet metal as follows: Blanks of the proper shape are first cut from the sheet metal. These are then pressed into the required form in a die. In computing the size of the blank, it is assumed that it has the same area as the finished article.



Find the diameter of the blank for the cover of a tin pail for holding lard, if the cover has a diameter of $6\frac{1}{2}$ in. and the flange is $\frac{5}{8}$ in.

19. A cylindrical metal box for holding paper fasteners is $1\frac{3}{4}$ in. in diameter and $1\frac{1}{2}$ in. deep, and the cover has a $\frac{1}{2}$-inch flange. Find the diameter of the blanks for box and cover. *Ans.* 3.68 in.; 2.56 in.

20. What is the locus of points 12 in. distant from two parallel lines 18 in. apart?

21. Holes for rivets are often punched in metal plates. The pressure required for such work, when the material is wrought iron, is 55,000 pounds per square inch of the cut surface in. For example, a hole having a circumference of 2 in. punched in a $\frac{1}{4}$-in. plate would require a pressure of $2 \times \frac{1}{4} \times 55{,}000$ lb., that is, the area of the cylindrical surface sheared off times 55,000 lb.

Find the pressure required to punch a $\frac{1}{4}$-in. round hole through a piece of sheet iron $\frac{1}{8}$-in. thick.

22. Find the pressure necessary to punch a $\frac{1}{2}$-inch round hole through a boiler plate $\frac{5}{8}$ in. thick if 60,000 pounds pressure is required per square inch of surface cut.

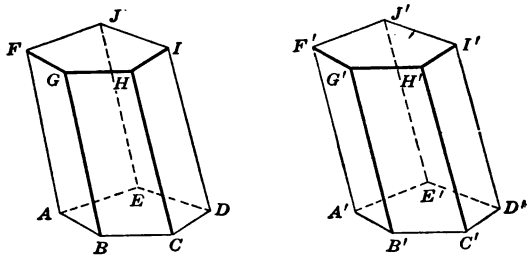
23. A rectangular box is to be made of sheet steel $\frac{1}{32}$ in. thick. Find the number of pounds the punch press must strike in order to cut out the blank if it is rectangular $5\frac{3}{4}$ in. by $2\frac{3}{4}$ in., has squares $\frac{7}{8}$ in. on a side cut from each corner, and has two holes $\frac{3}{16}$ in. in diameter punched in the bottom. Use 60,000 pounds per square inch. *Ans.* 34,084 lb.

CONGRUENT AND EQUIVALENT SOLIDS

654. Congruent Solids. Two solids are said to be **congruent** if they can be made to coincide completely in all their parts.

655. Equivalent solids are those having the same volume.

656. Theorem. *Two prisms are congruent if three faces including a triedral angle of one are congruent respectively to three faces including a triedral angle of the other, and are arranged in the same order.*



Given prism AI and $A'I'$, with face $AJ \cong$ face $A'J'$, face $AG \cong$ face $A'G'$, and face $AD \cong$ face $A'D'$.

To prove prism $AI \cong A'I'$.

Proof. $\angle BAF = \angle B'A'F'$, $\angle FAE = \angle F'A'E'$, and $\angle BAE = \angle B'A'E'$. Why?

Then triedral $\angle A \cong$ triedral $\angle A'$. § 610

Place the prism $A'I'$ upon prism AI so that triedral $\angle A'$ shall coincide with its congruent triedral $\angle A$.

Then face $A'J'$ coincides with its congruent face AJ; $A'G'$ with its congruent face AG; and $A'D'$ with its congruent face AD.

And points F', G', and J' fall upon F, G, and J respectively.

Further $C'H'$ coincides with CH. Why?

Hence H' falls upon H.

Similarly I' falls upon I.

$\therefore$ prism $AI \cong$ prism $A'I'$. § 654

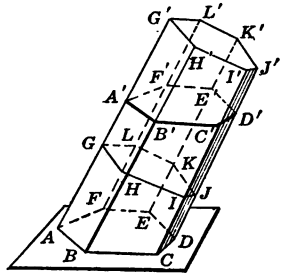
657. Theorem. *Two right prisms, or two right cylinders, are congruent if they have congruent bases and equal altitudes.*

658. Truncated Prism. A portion of a prism included between the base and a section oblique to the base is called a **truncated prism.**

659. Theorem. *Two truncated prisms are congruent if three faces including a triedral angle of one are congruent respectively to the three faces including a triedral angle of the other, and are arranged in the same order.*

660. Theorem. ***An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is a lateral edge of the oblique prism.***

Given the oblique prism AD'; also the right prism GJ' whose base GJ is a right section of the prism AD', and whose altitude is equal to a lateral edge AA' of prism AD'.



To prove prism AD' = prism GJ'.

Outline of proof. Show that $AG = A'G'$, $BH = B'H'$, etc.; that the angles of face AH equal the angles of face $A'H'$, and that face $AH \cong$ face $A'H'$.

Likewise show face $BI \cong$ face $B'I'$, and face $AD \cong$ face $A'D'$.

Then prism $AJ \cong$ prism $A'J'$. § 656

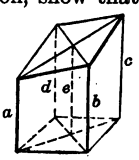
Prism AJ + prism GD' = prism $A'J'$ + prism GD'. Why?

$\therefore$ prism AD' = prism GJ'. Why?

EXERCISES

1. Compare the theorem of § 660 with the theorem of § 354.

2. If a wooden beam has a rectangular right cross section, show that if it is sawed lengthwise along a diagonal plane the two prisms formed are congruent.



3. In a truncated prism having a parallelogram for base, the sum of two opposite lateral edges is equal to the sum of the other two opposite lateral edges.

Prove $a + c = 2e$, and $b + d = 2e$.

PARALLELEPIPEDS

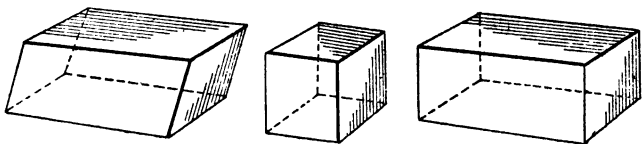
661. A **parallelepiped** is a prism whose bases are parallelograms.

662. A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to its bases.

663. An **oblique parallelepiped** is a parallelepiped whose lateral edges are oblique to its bases.

664. A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

665. A **cube** is a rectangular parallelepiped whose edges are all equal.



OBLIQUE PARALLELEPIPED CUBE RECTANGULAR PARALLELEPIPED

666. The **dimensions** of a rectangular parallelepiped are the lengths of the three edges drawn from one vertex. These dimensions are often called *length*, *breadth*, and *height*.

667. A **diagonal** of a parallelepiped is the line from any vertex to a vertex not in the same face.

668. The **volume** of any solid is the numerical measure of its magnitude in terms of some unit of measure.

The unit of measure usually taken is a cube one linear unit on an edge.

669. Ratio between Solids. By the ratio of one solid to another is meant the ratio of their numerical measures.

670. Prove the following facts about parallelepipeds:

(1) *All the faces (including the bases) of a parallelepiped are parallelograms.*

(2) *A parallelepiped is bounded by three pairs of parallel congruent parallelograms.*

(3) *A parallelepiped has three sets of four equal edges.*

(4) *Any two opposite faces of a parallelepiped may be taken as bases.*

(5) *All the faces of a rectangular parallelepiped are rectangles.*

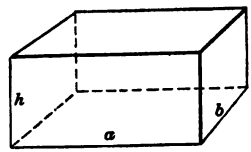
(6) *All the faces of a cube are squares.*

671. Volume of Rectangular Parallelepiped. By a consideration analogous to that of § 344 the reasonableness of the following statement would be evident; here it is accepted without proof.

The volume of a rectangular parallelepiped is equal to the product of its three dimensions.

If V, a, b, and h are the numerical measures of the volume, length, breadth, and height, respectively, of any rectangular parallelepiped, then the above is stated in the formula

$$V = abh.$$



This means that, if the three dimensions are each measured in terms of some unit of length, then the volume is measured in terms of the corresponding unit of volume, which is a cube having an edge one linear unit in length.

672. Theorem. ***The volume of a rectangular parallelepiped is equal to the product of its base and altitude.***

673. Theorem. *The volume of a cube is equal to the cube of its edge.*

674. Theorem. *The volumes of two rectangular parallelepipeds are to each other as the products of their three dimensions.* $V : V' = abh : a'b'h'$.

675. Theorem. *The volumes of two rectangular parallelepipeds having equivalent bases are to each other as their altitudes.*

Since $\frac{V}{V'} = \frac{abh}{a'b'h'}$ and $ab = a'b'$, then $\frac{V}{V'} = \frac{h}{h'}$.

676. Theorem. *The volume of two rectangular parallelepipeds having equal altitudes are to each other as their bases.*

677. Theorem. *Two rectangular parallelepipeds having equivalent bases and equal altitudes are equal in volume.*

EXERCISES

1. Find the volumes of the following rectangular parallelepipeds: (1) 13 ft. by 27 ft. by 45 ft. (2) 2 ft. 3 in. by 1 ft. 7 in. by 3 ft. 1 in. (3) 30 ft. 6 in. by 41 ft. 6 in. by 12 ft.

2. A common brick is 8 in. by 4 in. by 2 in. Find the number of common brick in a pile 3 ft. by 6 ft. by 12 ft.

3. How many shoe boxes each 3 in. by 4 in. by 9 in. can be put in a packing box 3 ft. by 3 ft. 4 in. by 3 ft. 9 in.?

4. Find the number of cubic yards of earth to be excavated in digging a cellar 40 ft. by 26 ft. by 7 ft.

5. If 180 sq. ft. of zinc are required to line the bottom and sides of a cubical vessel, how many cubic feet of water will it hold?

6. A box car that is $36\frac{2}{3}$ ft. long and 8 ft. wide, inside measurements, can be filled with wheat to a height of $4\frac{1}{2}$ ft. Find how many bushels of wheat it will hold if $\frac{5}{4}$ cu. ft. are a bushel.

7. Find the cost at 40 cents a pound for sheet copper to line the bottom and sides of a cubical vessel 7 ft. on an edge, if the sheet copper weighs 12 oz. per square foot. Find volume. *Ans.* \$73.50; 81.45 bbl.

8. Find the cost of laying a stone wall 45 ft. long, 6 ft. high, and 2 ft. thick at \$2.75 a perch. Use 22 cu. ft. for one perch.

9. The edge of a cube is 10 in. Find the edge of a cube which shall have a volume twice as great. Eight times as great.

10. Show that the edge and diagonal of a cube can be used as the two sides of a right triangle whose acute angles are 30° and 60°.

11. Are the diagonals of a rectangular parallelepiped perpendicular to each other? Are those of a cube? Prove.

12. The diagonals of a rectangular parallelepiped are equal.

13. The diagonals of a rectangular parallelepiped bisect each other.

14. Find the sum of all the face angles of a parallelepiped.

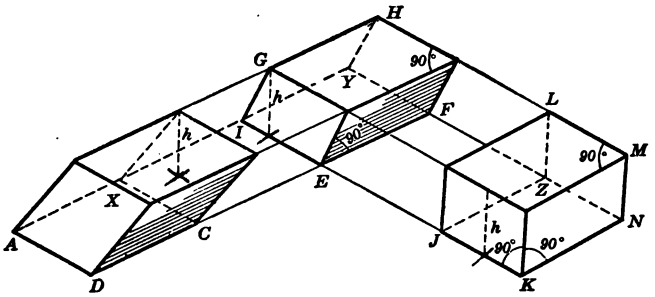
15. Find the edge of a cube if its volume is increased 200 cu. in. when each edge is increased 2 in.

16. The edges of a rectangular parallelepiped are a, b, and c. Find the length of a diagonal, and the entire area of the parallelepiped.

17. Test the following rule if 1 cu. in. of iron weighs 0.28 pounds: The weight of iron bars in pounds per foot of length equals the width in inches times the thickness in inches times $\frac{10}{3}$.

18. The sum of the squares of the four diagonals of any parallelepiped is equal to the sum of the squares of the twelve edges. Apply Ex. 11, p. 181.

678. Theorem. ***The volume of any parallelepiped is equal to the product of its base and altitude.*** $V = Bh$.



Given X an oblique parallelepiped, with area of base $AC = B$, and height h.

To prove $V = Bh$, where V denotes volume.

Proof. Extend DC and the other edges of X that are parallel to it.

On DC extended, take $EF = DC$.

Pass planes EG and FH through E and F and perpendicular to DC, forming a right parallelepiped Y, in which EG is a right section.

Then parallelepiped X = parallelepiped Y. § 660

Extend IE and the other edges of Y that are parallel to it.

On IE extended, take $JK = IE$.

Pass planes JL and KM through J and K and perpendicular to IE, forming a rectangular parallelepiped Z.

Then parallelepiped Y = parallelepiped Z. § 660

Therefore parallelepiped X = parallelepiped Z. § 104

But volume of Z = base $JN \times NM$. § 672

In which $NM = h$. Why?

And $\square JN = \square IF = \square AC = B$. Why?

$\therefore V = Bh$. Why?

679. Theorem. *Parallelepipeds having equivalent bases and equal altitudes are equivalent.*

680. Theorem. *Two parallelepipeds are to each other as the products of their bases and altitudes.*

681. Theorem. *Two parallelepipeds having equal altitudes are to each other as their bases.*

682. Theorem. *Two parallelepipeds having equivalent bases are to each other as their altitudes.*

EXERCISES

1. Compare theorems of §§ 678-682 with those of §§ 355, 358-361.

2. Use the notation of § 678, and show that in any parallelepiped $B=\frac{V}{h}$, and $h=\frac{V}{B}$.

3. Find the length of the diagonal of a rectangular parallelepiped whose dimensions are 30 ft., 40 ft., and 12 ft.

4. A parallelepiped has a base in the form of a rhombus whose edges and one diagonal are each 10 in. Find the volume if the altitude is 8 in.

5. A parallelepiped has a rectangular base 8 in. by 15 in., and square ends. Find its volume if one of the sides is a parallelogram having an angle of 60°.

6. Find the diagonal of a cube whose volume is 512 cu. in.

7. Find the dimensions of a rectangular parallelepiped having a volume of 12,960 cu. in., if the dimensions are in the ratio of 3 : 4 : 5.

Ans. 18 in., 24 in., 30 in.

8. Find the edge of a cube whose surface and volume have the same numerical value.

9. A parallelepiped of altitude 8 in. has the same volume as a cube that is 12 in. on an edge. Find the area of the base and the dimensions of the base if it is a parallelogram having an angle of 60° and one side equal to 18 in.

10. Find the length of the bar that can be made from 1 cu. ft. of steel, if the bar has a rectangular cross section $\frac{1}{2}$ in. by $1\frac{1}{2}$ in.

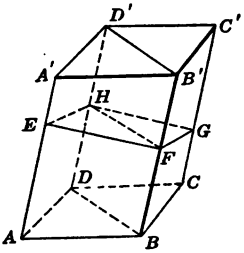
11. Find the volume to the nearest .001 cu. ft. of a rectangular tank 17 ft. 2 in. by 19 ft. 3 in. by 3 ft. 7 in. *Ans.* 1184.142 cu. ft.

12. Show geometrically that $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

13. The total area of a right prism whose base is a rectangle is 118 sq. ft., the volume is 70 cu. ft., and the altitude is 7 ft. Find dimensions of the base.

VOLUMES OF PRISMS AND CYLINDERS

683. Theorem. *The plane passed through the two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.*



Given the parallelepiped AC', divided into two triangular prisms $A'\text{-}ABD$ and $C'\text{-}BCD$ by the diagonal plane passing through the edges BB' and DD'.

To prove prism $A'\text{-}ABD$ = prism $C'\text{-}BCD$.

Proof. Let $EFGH$ be a right section of the parallelepiped. cutting the diagonal plane in FH.

Face $AB' \parallel$ face DC', and face $AD' \parallel$ face BC'.	Why?
Then $EF \parallel HG$, and $EH \parallel FG$.	§ 562
And $EFGH$ is a parallelogram.	Why?
FH is a diagonal of $\square EFGH$.	Why?
Hence $\triangle EFH \cong \triangle FGH$.	§ 154

Prism $A'\text{-}ABD$ equals a right prism with base EFH and altitude AA'. § 660

Prism $C'\text{-}BCD$ equals a right prism with base FGH and altitude AA'. Why?

But these two right prisms are congruent and so equivalent.

$\therefore$ prism $A'\text{-}ABD = C'\text{-}BCD$. § 104

EXERCISES

1. Find the edge of a cube that is increased in volume 127 cu. in. when its edge is increased 1 in.

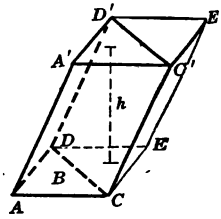
2. A rectangular solid with a square base has a volume of 80 cu. ft., and a surface of 112 sq. ft. Find its dimensions. *Ans.* 4 ft. by 4 ft. by 5 ft.

684. Theorem. ***The volume of a triangular prism is equal to the product of its base and altitude.*** $V = Bh$.

Given the triangular prism A'-ACD.

To prove $V = Bh$, where V denotes volume, B area of base, and h altitude.

Suggestion. Complete the parallelepiped A'-$ACED$. Show that prism A'-$ACD = \frac{1}{2}$ parallelepiped A'-$ACED$. Apply § 683.

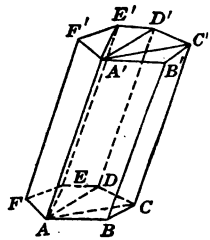


685. Theorem. *The volume of any prism is equal to the product of its base and altitude.* $V = Bh$.

Given any prism AD'.

To prove $V = Bh$, where V denotes volume, B area of base, and h altitude.

Suggestion. Show that the prism can be divided into triangular prisms by diagonal planes. Add the prisms thus formed.



686. Theorem. *Prisms having equivalent bases and equal altitudes are equivalent.*

687. Theorem. *Prisms having equivalent bases are to each other as their altitudes.*

688. Theorem. *Prisms having equal altitudes are to each other as their bases.*

EXERCISES

1. Find the volume of a prism whose base is a regular hexagon 6 in. on a side, and whose altitude is 24 in.

2. In a prism given $V = 226$ and $B = 43.6$; find h.

3. The sides of a right section of a triangular prism are 4 in., 5 in., and 7 in. Find the volume if a lateral edge is 16 in.

4. The cost of digging a ditch, including all expenses and profits, is estimated at 27 cents a cubic yard. Find the cost of digging a ditch 15 mi. long, 10 ft. wide at the bottom, 20 ft. at the top, and 6 ft. deep. The cross section is a trapezoid. *Ans.* $71,280.

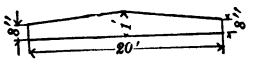
5. Show that an oblique prism of wood may be changed into a right prism by cutting along a right section and interchanging the two parts.

6. One of the concrete pillars to support a floor in a concrete building is 12 ft. high and has as a cross section a regular hexagon 8 in. on a side. Find its weight if the concrete weighs 138 lb. per cubic foot. *Ans.* 1912 lb.

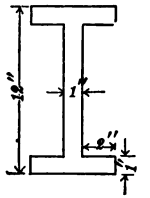
7. A flow of 300 gallons per second will supply water for a stream of what depth, if the stream is 4 ft. wide and flows 5 miles per hour?

Ans. $16\frac{1}{3}\frac{3}{2}$ in.

8. Find the number of cubic yards of crushed rock to make a road 1 mile in length and of cross section as shown in the figure.



9. One cubic inch of steel weighs 0.29 lb. An I-beam has a cross section as shown in the figure and a length of 22 ft.. Find its weight.



10. An iron casting shrinks $\frac{1}{8}$ in. per linear foot in cooling down to 70 degrees Fahrenheit. How many cubic inches is the shrinkage per cubic foot?

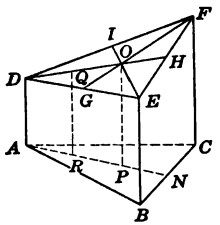
Ans. 53.44 cu. in.

11. How many cubic yards of soil will it take to fill in a lot 50 ft. by 100 ft. if it is to be raised 3 ft. in the rear and gradually sloped to the front where it is to be $1\frac{1}{2}$ ft. deep? *Ans.* $416\frac{2}{3}$ cu. yd.

12. In a regular triangular prism, the edge of the base and the altitude are equal. Find these dimensions if the volume is $128\sqrt{3}$ cu. in.

13. In a regular triangular prism, the altitude is 6 in. more than the edge of the base. Find the dimensions if the volume is $224\sqrt{3}$ cu. in.

14. Find the volume of a regular hexagonal prism whose base has an area of 37.5 sq. ft., and whose altitude equals an edge of the base.



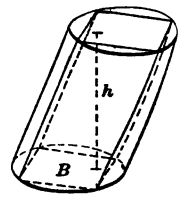
15. The perpendicular drawn to the lower base of a truncated triangular prism from the intersection of the medians of the upper base, equals one third the sum of the three lateral edges.

16. The volume of any oblique prism is equal to the product of the area of a right section by the length of a lateral edge.

17. The volume of a regular hexagonal prism is $30\sqrt{3}$ cu. in. and its lateral area is 180 sq. in. Find its altitude and base edge.

18. The volume of a triangular prism is equal to the area of a lateral face times one-half the perpendicular drawn to that face from the opposite edge.

689. Theorem. ***The volume of a circular cylinder is equal to the product of its base and altitude.*** $V = Bh$.



Given a circular cylinder.

To prove that $V = Bh$, where V denotes volume, B area of base, and h altitude.

Proof. Inscribe a prism, whose base is a regular polygon, in the cylinder, and let V' denote its volume and B' the area of its base.

By doubling indefinitely the number of faces of the prism,

$V' \to V$, and $B' \to B$. §§ 651 (3), 489

Then $B'h \to Bh$. § 485 (2)

But $V' = B'h$, being variables that are always equal. § 685

$\therefore V = Bh$. § 485 (1)

EXERCISES

1. Show that the volume of a hollow cylinder with outer diameter D, inner diameter d, and altitude h, is given by the formula,

$$V = \tfrac{1}{4}\pi h(D^2 - d^2) = \tfrac{1}{4}\pi h(D+d)(D-d).$$ (See Ex. 8, p. 249.)

2. A cylindrical oil tank 3 ft. in diameter and 10 ft. long will contain how many gallons? (1 gal. = 231 cu. in.) *Ans.* 528.8.

3. The outer diameter of a hollow cast iron shaft is 18 in.; and its inner diameter is 10 in. Calculate its weight if the length is 20 ft. and cast iron weighs 0.26 lb. per cubic inch.

4. A peck measure is to have a diameter of 8 in. How deep should it be? (1 bu. = 2150.42 cu. in.)

5. Water is flowing at the rate of 10 miles per hour through a pipe of diameter 16 in. into a rectangular reservoir 197 yd. long and 87 yd. wide. Calculate the time in which the surface of the water in the tank will be raised 3 in. *Ans.* 31.38 minutes.

6. A certain handbook gives the following "rules of thumb" for finding the volume in gallons of a cylindrical tank:

(1) V = (diameter in feet)$^2 \times 5\frac{7}{8} \times$ (height in feet).

(2) V = (diameter in feet)$^2 \times \frac{1}{2}$ (height in inches) less 2% of the product.

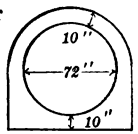
Find the per cent of error for each rule.

7. Find the height of a 10-gallon wash boiler whose base is 10 in. wide with semicircular ends, the length of the straight part of the sides being $9\frac{1}{4}$ in. *Ans.* 13.5 in.

8. A cylinder to cool lard is 4 ft. in diameter and 9 ft. long and makes four revolutions per minute. At each revolution, the hot lard cools upon the surface to a depth of $\frac{1}{8}$ in. How many pounds of lard will it cool in one hour if 1 cu. ft. of lard weighs $56\frac{1}{4}$ pounds?

9. If a tank 5 ft. in diameter and 10 ft. deep holds 10,000 pounds of lard, what will be the depth of a tank of 2000 pounds capacity if its diameter is 3 ft.? If this tank has a jacket around it on the bottom and sides 3 in. from the surface of the tank, how many gallons of water will the space between the jacket and the tank hold?

10. A conduit made of concrete has a cross section with dimensions as shown in the figure. How many cubic yards of concrete are used in making one mile of this conduit?



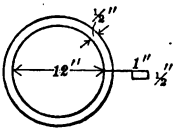
11. The cylindrical water tower shown in the figure is at Long Beach, N.Y. Its diameter is 34 ft., height 150 ft., and it is said to have a capacity of 1,020,000 gallons. Is the capacity given correct?

12. Does a cylindrical water tank 42 in. in diameter and 14 ft. long hold 1000 gallons?

13. The following "rule of thumb" is used for finding the weight of round iron. The weight of round iron in pounds per foot equals the square of the diameter in quarter inches divided by 6. Find the per cent of error in using the rule if iron weighs 0.28 lb. per cubic inch.

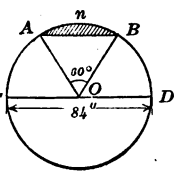
14. How much water will a horizontal steam boiler 5 ft. in diameter and 16 ft. long with 70 tubes of diameter 3 in. running lengthwise, hold if one third of the volume is for steam?

15. Find the height of a cylindrical oil tank with a diameter of 16 in. to hold one barrel. *Ans.* 36.2 in.



16. A tool steel ring for a steam cylinder is forged from round stock 3 in. in diameter. Find the length of stock to make a ring with the dimensions given in the figure.

17. The segment in the figure is a counter-balance $5\frac{1}{2}$ in. thick. Find its weight if made of cast-iron weighing 0.26 lb. per cubic inch. *Ans.* 228.5 lb.



18. Find the weight of a hollow hexagonal bar 16 ft. long and weighing 0.28 lb. per cubic inch. The cross section is a regular hexagon $1\frac{1}{4}$ in. on a side, with a circle $1\frac{1}{4}$ in. in diameter at the center.

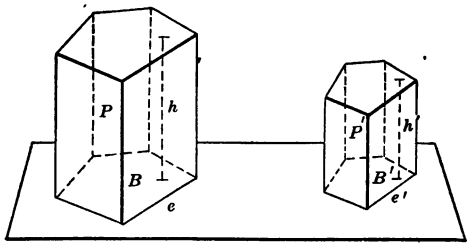
Ans. $152\frac{1}{4}$ lb.

SIMILAR PRISMS AND CYLINDERS

690. Definitions. Two right prisms having similar polygons for bases, and whose altitudes are in the same ratio as two corresponding base edges, are **similar.**

Two right circular cylinders are **similar** if their altitudes are in the same ratio as the radii of their bases.

691. Theorem. *The volumes of two similar right prisms are in the same ratio as the cubes of corresponding base edges.*



Given two similar right prisms P and P', having altitudes h and h', bases B and B', and two corresponding base edges e and e'

To prove $\frac{V}{V'}=\frac{h^3}{h'^3}=\frac{e^3}{e'^3}$.

Proof.

$V=Bh$, and $V'=B'h'$. § 685

Hence $\frac{V}{V'}=\frac{Bh}{B'h'}=\frac{B}{B'}\cdot\frac{h}{h'}$. § 108

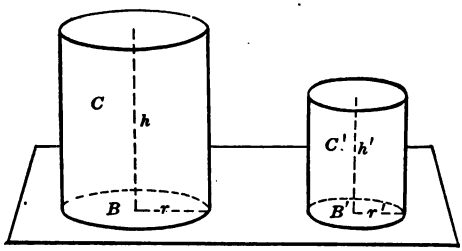
But $\frac{B}{B'}=\frac{e^2}{e'^2}$. § 446

And $\frac{h}{h'}=\frac{e}{e'}$. § 690

$\therefore \frac{V}{V'}=\frac{e^2}{e'^2}\cdot\frac{e}{e'}=\frac{e^3}{e'^3}$. § 111

Also $\frac{V}{V'}=\frac{h^3}{h'^3}$. § 111

692. Theorem. *The volumes of two similar right circular cylinders are in the same ratio as the cubes of their altitudes, radii, or diameters.*



Given two similar right circular cylinders C and C', having volumes V and V', bases B and B', altitudes h and h', radii r and r', and diameters d and d'.

To prove $\frac{V}{V'}=\frac{h^3}{h'^3}=\frac{r^3}{r'^3}=\frac{d^3}{d'^3}$.

The proof is similar to the preceding.

EXERCISES

1. Using the notation of § 691 for similar prisms, also S and S' for lateral areas, and T and T' for total areas, prove the following:

(1) $\frac{S}{S'}=\frac{e^2}{e'^2}=\frac{h^2}{h'^2}$. (2) $\frac{T}{T'}=\frac{e^2}{e'^2}=\frac{h^2}{h'^2}$.

2. Using the notation of § 692 for similar cylinders, also S, S', T, and T' as in Exercise 1, prove the following:

(1) $\frac{S}{S'}=\frac{h^2}{h'^2}=\frac{r^2}{r'^2}=\frac{d^2}{d'^2}$. (2) $\frac{T}{T'}=\frac{h^2}{h'^2}=\frac{r^2}{r'^2}=\frac{d^2}{d'^2}$.

3. Find the ratios of the volumes, the lateral areas, and the total areas of two similar right prisms having hexagonal bases two of whose corresponding edges are 2 in. and 5 in. respectively.

4. Two similar right circular cylinders have radii of 3 in. and 7 in. respectively. Find the ratio of their volumes. Of their lateral areas. Of their total areas.

5. Two similar right circular cylinders have bases whose areas are 20.25 sq. in. and 100 sq. in. respectively. Find the ratio of their volumes.

6. The total areas of two similar right circular cylinders are 625 sq. in. and 324 sq. in. respectively. Find the diameter of the second if the diameter of the first is $6\frac{1}{4}$ in.

7. A rectangle having dimensions of 10 in. and 15 in. is revolved, first about the side that is 10 in., and second, about the side that is 15 in. Find the ratio of the volumes of the two cylinders formed.

8. The number of feet of lumber in a log is often based upon a standard log usually 12 ft. long and 24 in. in diameter inside the bark at the small end. If v, d, and l are the volume, diameter, and length respectively of the standard log; and V, D, and L, the corresponding measurements of the log to be measured, then $V = \frac{vD^2L}{d^2l}$. Derive this formula.

QUESTIONS

1. What is a prismatic surface? A cylindrical surface? Is the directrix necessarily closed?

2. In § 617, suppose the directrix is coplanar with the generatrix, what is the form of the surface generated?

3. What are the formulas or rules for finding areas of prisms and cylinders? What are the formulas or rules for finding the volumes of each of these solids?

4. Why is there such a close relation between theorems concerning prisms and cylinders?

5. Can you find the area of an oblique circular cylinder? Of an oblique prism? Can you find the volume of each of these?

6. Trace the steps in finding the volume of a parallelepiped.

7. Trace the steps in finding the volume of any prism. Of a cylinder.

8. What effect does it have upon the volume of a prism or cylinder if the base is doubled? If the altitude is doubled? If both base and altitude are doubled?

9. What effect does it have upon the lateral area of a right circular cylinder if the circumference of the base is doubled? If the area of the base is doubled?

10. Which do you consider the more common form in nature, the cylinder or the prism? Name some examples of each form.

11. Which is the more common of these forms in buildings and architecture? In machinery? Give illustrations of triangular prisms, of quadrangular prisms, of pentagonal, of hexagonal, etc.

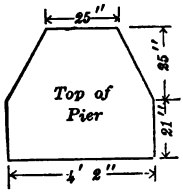
GENERAL EXERCISES

COMPUTATIONS

1. Find the number of barrels each of the following cylindrical tanks will hold: (1) diameter 5 ft. and depth 5 ft., (2) diameter 20 ft. and depth 19 ft. (1 bbl. = 3.211 cu. ft.)

2. Find the cost of laying the stonework, at \$1.75 per cubic yard, in two abutments for a bridge, each abutment to be 8 ft. high, $3\frac{1}{2}$ ft. thick, 20 ft. long at the bottom, and 15 ft. at the top.

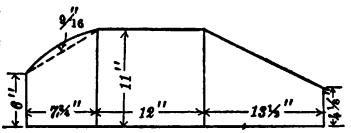
3. Find the cost of common brick in the pier with a cross section as shown in the figure, and a height of 12 ft. 6 in., at \$7.00 per thousand. Count 20 brick to 1 cu. ft. *Ans.* \$24.15.



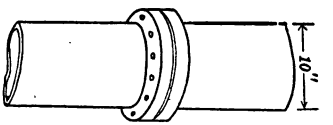
4. A stream flowing 5 miles per hour must be how large in cross section to supply water to a depth of $1\frac{1}{2}$ in. per week, for 160 acres of land? *Ans.* $28\frac{2}{7}$ sq. in.

5. A water tank in a Pullman car has a vertical section as shown in the figure, and a length of 52 in. Find its capacity in gallons. *Ans.* 68.3 gal.

Consider the arc as a part of a circle and apply formula (1) of § 509.



6. The flanges at the joining of two ends of flanged steam pipes 10 in. in inside diameter are bolted together by 14 bolts $\frac{3}{4}$ in. in diameter. If the pressure in the pipes is 200 pounds per square inch, find what each bolt must hold. How much is this per square inch cross section of the bolts? Suppose that the bolts have 10 pitch U.S. standard thread. This makes the root diameter 0.620 in.



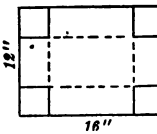
Ans. 1122 lb.; 3718— lb.

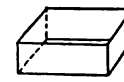
7. Find the diameter of a cylindrical oil tank 40.5 in. high that is to hold 1 barrel. *Ans.* $15\frac{1}{8}$ in.

8. Representing the dimensions of a rectangular solid by x, y, and z, find their values if when each is increased 2 in. the volume is increased 150 cu. in., the face of dimensions x and y is increased 18 sq. in., and the total surface is increased 110 sq. in. *Ans.* 3 in., 4 in., and 5 in.

9. Two cubes whose edges differ by 1 in. have volumes that differ by 397 cu. in. Find the edges of each cube.

10. A rectangular sheet of tin, 12 in. by 16 in., is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cu. in.





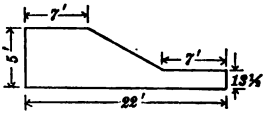
11. A rectangular piece of tin a inches longer than it is wide is made into an open box, containing c cubic inches, by cutting from each corner a square of side b inches. Find the dimensions of the original piece of tin.

12. One edge of a rectangular box is increased 6 in., another 3 in., and the third 4 in., making a cube whose volume is 862 cu. in. greater than that of the original box. Find the dimensions of the box.

Ans. 4.14 in. by 7.14 in. by 6.14 in.

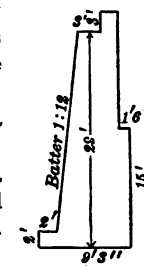
13. If in a right prism the altitude is equal to a side of the base, find the volume if the base is an equilateral triangle whose sides are a.

14. Find the capacity in gallons of a water tank for a locomotive tender. The dimensions are as given in the figure for the length cross section, and the width is 9 ft. 6 in.



15. A certain coal and coke company finds it necessary to construct a wharf wall 150 ft. long, and having a cross section with dimensions as shown in the figure. Find the number of cubic yards in the wall.

By "batter 1:12" is meant that the slant is 1 ft. in a vertical rise of 12 ft.



16. In a right prism whose volume is 54, the lateral area is four times the area of the base which is an equilateral triangle. Find the edge of the base. *Ans.* 6.

17. The total areas of two similar cylinders of revolution are 75 sq. in. and 192 sq. in. respectively. If the volume of the first cylinder is 250 cu. in., what is the volume of the second?

18. In a right circular cylinder, given $V=\pi r^2 h$ and $T=2\pi r^2+2\pi rh$. Find T in terms of V and h.

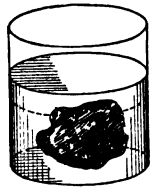
Ans. $T=\dfrac{2V}{h}+2\sqrt{V\pi h}$.

19. In a right circular cylinder, find V in terms of the circumference c of the base and the total area T.

Ans. $V=\dfrac{2\pi cT-c^3}{8\pi^2}$.

20. Find the steam capacity of a horizontal cylindrical boiler 4 ft. in diameter and 16 ft. long, if the height of the segment occupied by the steam is 18 in. (Use (2) of §509).

21. A cylindrical tank of diameter 30 in. and 34 in. long rests on its side. Find the number of gallons of gasoline in the tank if the depth is $5\frac{1}{2}$ in. *Ans.* 13 gal. nearly.



22. The volume of an irregular shaped body is often found by immersing it in water and determining the amount of water displaced. A cylindrical vessel that has a diameter of 4 in. is partly filled with water. A stone immersed in the water raises its level $3\frac{1}{2}$ in. Find the volume of the stone.

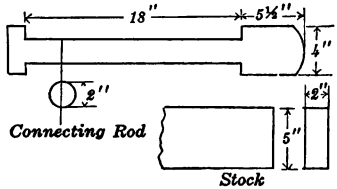
23. A silo is used to keep fodder in a green and succulent state for feeding farm animals. It is usually built in a cylindrical form.

Find the capacity in tons of a silo in the form of a right circular cylinder 20 ft. in diameter and 32 ft. high, if a cubic foot of silage weighs 40.7 lb.

24. An 18 acre field yields 11.5 tons of silage per acre. What must be the height of a cylindrical silo 20 ft. in diameter to hold all the silage if 1 cu. ft. of silage weighs 38.4 lb.?

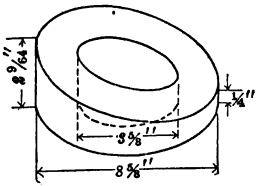
25. A silo is in the form of a right circular cylinder, and is 20 ft. in diameter inside and 32 ft. high. How many cubic yards of concrete did it take to build it, if the floor and wall are each 6 in. thick, and the foundation wall is 8 in. thick and 5 ft. deep?

26. The connecting rod with dimensions as given in the figure is made from stock with the dimensions shown. Find length of stock that it is necessary to allow for the cylindrical part of the rod.



27. Find the volume of the beveled washer with dimensions as shown in the figure.

28. A pine log 2 ft. in diameter and 16 ft. long is floating in water. Find the weight of the log if two-thirds of the volume of the log is under water. (Water weighs 62.5 lb. per cubic foot.)



29. Will a floating pine log $1\frac{1}{2}$ ft. in diameter and 10 ft. long support a man weighing 180 pounds if the specific gravity of the log is 0.72?

30. The following is the record of a test in which a high speed drill removed 70.56 cu. in. of cast-iron per minute. The penetration per minute was $57\frac{1}{2}$ in., the feed $\frac{1}{10}$ in. per turn, and the number of turns per minute was 575. Do these numbers agree?

THEOREMS AND PROBLEMS

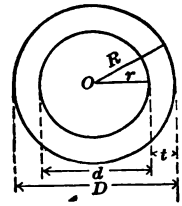
1. Derive the following formulas for finding the volume, V, of a hollow circular cylinder of length h and cross sectional dimensions as given in the figure:

(1) $V = \pi h(R+r)(R-r)$.

(2) $V = \frac{1}{2}\pi ht(D+d)$.

(3) $V = \pi ht(d+t)$.

(4) $V = \pi ht(D-t)$.



2. Construct a plane through a point and tangent to a given right circular cylinder.

3. If the radius of one cylinder is equal to the altitude of a second, and the radius of the second is equal to the altitude of the first, what is the ratio of their volumes?

4. The intersection of two planes each tangent to a circular cylinder is parallel to the elements of the cylinder.

5. Prove that the volume of a right circular cylinder is equal to its lateral area times one-half the radius of its base.

6. The volume of two right circular cylinders are equal. Write a proportion between their lateral areas and their radii.

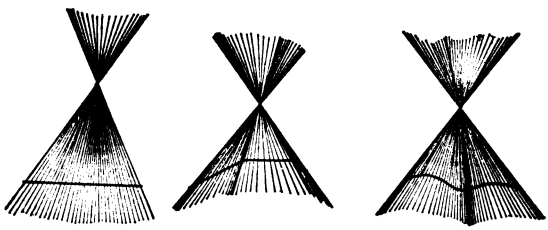
7. If a straight line has more than two points common to the curved surface of a right circular cylinder, the line is an element of the surface.

CHAPTER VIII

PYRAMIDS AND CONES

693. A moving straight line that always contains a fixed point, and always intersects a given straight line, generates a **plane.** Why? § 542

694. Pyramidal Surface. A moving straight line that always contains a fixed point, and always intersects a broken line not coplanar with it, generates a **pyramidal surface.**



695. Conical Surface. A moving straight line that always contains a fixed point, and always intersects a plane curved line not coplanar with it, generates a **conical surface.**

696. The fixed point is called the **vertex** of the pyramidal, or conical, surface.

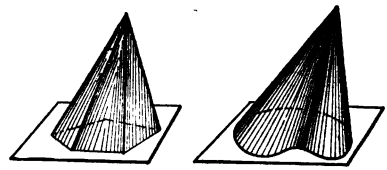
697. The two parts of the pyramidal, or conical, surface on opposite sides of the vertex are called **nappes.**

The words *generatrix*, *directrix*, *element*, and *closed surface* have the same significance here as in §§ 618, 619.

The directrix is not necessarily closed, but in this text only those that are closed are considered.

The word "section" has the same significance as before (§ 620), however, the cutting plane must not pass through the vertex.

698. A **pyramid** is the solid formed by cutting all the elements of one nappe of a closed pyramidal surface by a plane.



699. A **cone** is the solid form by cutting all the elements of one nappe of a closed conical surface by a plane.

The meanings of the words: *base, lateral face, lateral edge, base edge, lateral area, total area,* are apparent from the definitions of §§ 623, 624.

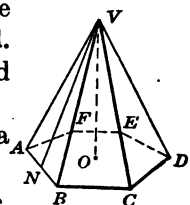
700. The **altitude** of a pyramid, or cone, is the perpendicular distance from the vertex to the base.

701. Pyramids Classified According to the Number of Lateral Faces. Pyramids, like prisms, are classified as **triangular, quadrangular, pentagonal,** etc., according as their bases are *triangles, quadrilaterals, pentagons,* etc.

702. Regular Pyramids. A pyramid whose base is a regular polygon, and whose altitude meets the center of its base, is called a **regular pyramid.**

The altitude of a regular pyramid is called its **axis.**

The altitude of one of the lateral faces of a regular pyramid is called the **slant height.**



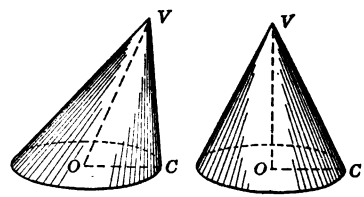
REGULAR PYRAMID

In the figure, OV is the altitude, BV a lateral edge, and NV the slant height.

703. Circular Cones. A cone whose base is a circle is called a **circular cone.**

The line joining the vertex of a circular cone to the center of its base is called the **axis** of the cone.

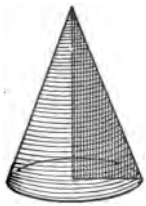
704. A **right circular cone** is a circular cone whose axis is perpendicular to its base.



CIRCULAR CONE RIGHT-CIRCULAR CONE

The radius of a circular cone is the **radius** of its base.

Since a right circular cone may be generated by revolving a right triangle about one of its sides as an axis it is often called a **cone of revolution.**



The length of an element of a right circular cone is its **slant height.**

705. Prove the following facts concerning cones and pyramids:

(1) *The lateral faces of a regular pyramid are congruent isosceles triangles.*

(2) *The lateral edges of a regular pyramid are equal.*

(3) *The elements of a right circular cone are equal.*

(4) *The axis of a right circular cone coincides with its altitude.*

(5) *A straight line drawn from the vertex of a cone to any point in the perimeter of its base is an element.*

(6) *The section of a circular cone made by a plane containing an element is a triangle.*

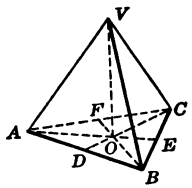
(7) *The section of a pyramid made by a plane through its vertex is a triangle.*

EXERCISES

1. Find an element of a right circular cone whose altitude is 15 and radius 7.

2. Find the altitude of a right circular cone whose slant height is s and radius r.

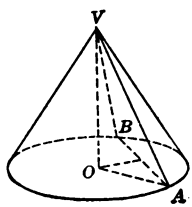
3. Find the altitude of a regular pyramid each face and the base being an equilateral triangle 10 in. on a side.



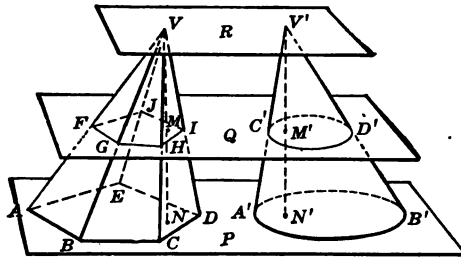
4. Into how many parts do the nappes of a conical surface divide space?

5. Why is it stated in the definition of a pyramidal surface that the vertex and directrix must not be coplanar?

6. The altitude of a right circular cone is 10 in. and the radius of the base is 6 in. Find the area of a section made by a plane passing through the vertex and 3 in. from the center of the base. In the figure find the area of VAB.

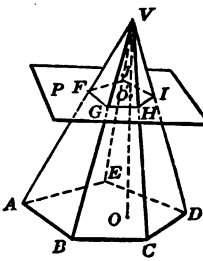


706. Theorem. *The lateral edges and the altitude of a pyramid, or the elements and the altitude of a cone, are divided proportionally by a plane parallel to its base.*



Pass a plane through the vertex parallel to the base and apply § 568.

707. Theorem. *The section of a pyramid made by a plane parallel to the base is similar to the base.*



Given pyramid $V\text{-}AD$, cut by plane P parallel to base AD, forming the section FI.

To prove polygon $FI \sim$ polygon AD.

Proof. $\triangle FVG \sim \triangle AVB$, $\triangle GVH \sim \triangle BVC$, etc. Why?

Then $\frac{FG}{AB} = \frac{VG}{VB}$, and $\frac{GH}{BC} = \frac{VG}{VB}$, etc. § 428

Hence $$\frac{FG}{AB} = \frac{GH}{BC} = \frac{HI}{CD} = \frac{IJ}{DE} = \frac{JF}{EA}.$$ Why?

Also $\angle JFG = \angle EAB$, $\angle FGH = \angle ABC$, etc. § 567

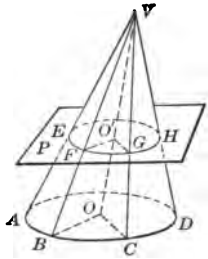
$\therefore$ polygon $FI \sim$ polygon AD. §§ 441, 431

EXERCISES

1. The lateral edges of a pyramid are respectively 15 in., 12 in., and 16 in. Find the parts of each made by a plane that is parallel to the base and which divides the altitude into parts that are in the ratio of 1 : 3.

2. The area of the base of a pyramid is 125 sq. in. Find the area of a section 8 in. from the base and parallel to it, if two corresponding edges of the base and the section are respectively 10 in. and 8 in.

708. Theorem. *The section of a circular cone made by a plane parallel to its base is a circle, the center of which is the point where the axis intersects it.*



Given a circular cone V-ABC, with section EH made by the plane P parallel to the base. Also given axis VO piercing plane P in O'.

To prove that EH is a circle having O' as center.

Proof. Take F and G any two points in EH.

Then the planes determined by $FO'V$ and $GO'V$ intersect the plane of the base in OB and OC, the plane P in $O'F$ and $O'G$, and the lateral surface in the elements VB and VC, respectively. Why?

Prove that $\triangle FO'V \backsim \triangle BOV$, and $\triangle GO'V \backsim \triangle COV$.

Then $\dfrac{FO'}{BO} = \dfrac{GO'}{CO}$. Why?

But $BO = CO$. Why?

Therefore $FO' = GO'$ Why?

Since F and G are any two points in the section EH, all points in this section are equally distant from O'.

$\therefore$ section EH is a circle with center O'. § 263

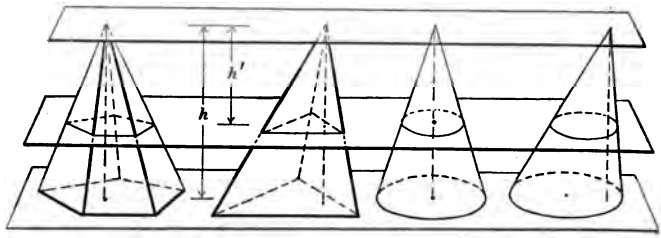
709. Theorem. *The area of any section of a pyramid, or circular cone, parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid, or cone.*

EXERCISES

1. The area of the base of a pyramid is 64 sq. in. and its altitude is 20 in. Find the area of a section parallel to the base and 4 in. from the vertex.

2. The diameter of the base of a circular cone is 6 in. and the altitude is 10 in. Find the area of a section parallel to the base and 6 in. from the vertex.

710. Theorem. *If two pyramids, two circular cones, or a pyramid and a circular cone, have equal altitudes and bases of equal area, sections made by planes parallel to the bases and at equal distances from the vertices, have equal areas.*



Suggestion. Let the equal altitudes be h and the distance from the vertices to the sections h'.

Then area of any section is to area of base as $h'^2 : h^2$. § 709

But it is given that the bases are equal, and the altitudes equal.

Therefore the sections are equal in area. Why?

EXERCISES

1. In the figure of § 707, $VO' = 8$, $O'O = 12$. $VB = 25$, and $VA = 27$; find VG and VF.

2. If the base of the pyramid of § 707 has an area of 180 sq. in., using the lengths given in Exercise 1, find the area of the section.

3. The area of the base of a pyramid is 150 sq. cm. and its altitude is 2 dm. Find the area of a section 8 cm. from the vertex.

4. The areas of parallel sections of a pyramid, or of a circular cone, are to each other as the squares of their distances from the vertex.

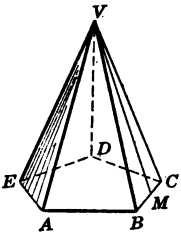
5. The altitude of a pyramid is h. At what distance from the vertex must a plane be passed parallel to the base so that the section shall be (1) one-half as large as the base? (2) One-third? (3) One-fifth?

6. Each side of the base of a regular hexagonal pyramid is 8 in. If the altitude of the pyramid is 20 in., how far from the vertex must a plane parallel to the base be so that the area of the section shall be $24\sqrt{3}$ sq. in.?

7. The area of the base of a cone is 216 sq. in. and its altitude is 24 in. Find the distance between two sections parallel to the base, which have areas of 144 sq. in. and 72 sq. in., respectively.

AREAS OF PYRAMIDS AND CONES

711. Theorem. ***The lateral area of a regular pyramid is equal to half the product of the perimeter of the base and the slant height.*** $S=\frac{1}{2}ps$.



Given the regular pyramid $V\text{-}ABCDE$.

To prove $S=\frac{1}{2}ps$, where S denotes lateral area, s slant height, and p perimeter of base.

Proof. The lateral faces are congruent triangles. Why?

The area of each face triangle $=\frac{1}{2}s\times$ its base. Why?

The sum of the bases of the triangles $=p$. Why?

Hence the sum of the areas of the faces $=\frac{1}{2}ps$. Why?

$\therefore S=\frac{1}{2}ps$.

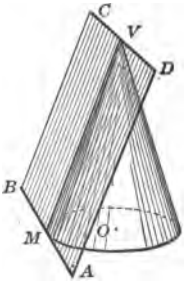
EXERCISES

1. Using the notation of §711 show that $s=\dfrac{2S}{p}$, and $p=\dfrac{2S}{s}$.

2. Find the lateral area and the total area of a regular hexagonal pyramid whose altitude is 8 in. and base edge is $4\sqrt{3}$ in.

3. Find the lateral area and the total area of a regular triangular pyramid having base edges equal to 8 in. and lateral edges equal to 12 in.

712. Plane Tangent to a Cone. A plane is **tangent to a cone** if it meets the conical surface in one element only.



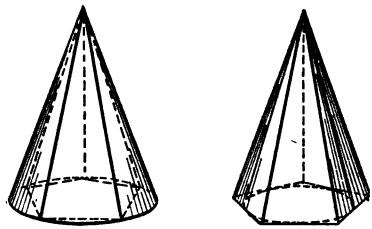
713. Theorem. *If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.*

Proof similar to that of § 642.

714. Theorem. *The plane determined by the tangent to the base of a circular cone and the element through the point of contact, is tangent to the cone.*

Compare with theorem of § 643.

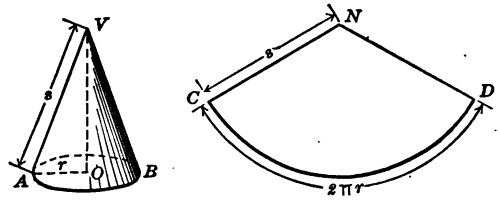
715. Inscribed and Circumscribed Pyramids. A pyramid is said to be **inscribed in a cone** if the pyramid and cone have the same vertex, and if the base of the pyramid is inscribed in the base of the cone. The cone is then said to be **circumscribed** about the pyramid.



716. A pyramid is said to be **circumscribed about a cone** if the pyramid and cone have the same vertex, and if the base of the pyramid is circumscribed about the base of the cone. The cone is then said to be **inscribed** in the pyramid.

It is evident that the lateral edges of an inscribed pyramid are elements of the cone; and that the lateral faces of the circumscribed pyramid are tangent to the cone by §§ 712, 715.

717. Area of Cone, Practical Consideration. If the lateral surface of a right circular cone is covered with paper, this paper when peeled from the cone can be spread out and will be a sector of a circle, whose radius is the slant height of the cone, and

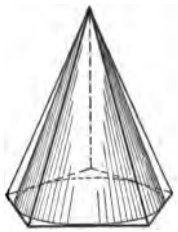


whose arc is equal to the circumference of the base of the cone. Then, by § 508, *the lateral area of a right circular cone is equal to half the product of the slant height and the circumference of the base.* $S=\frac{1}{2}sc=\pi rs$.

718. Geometric Treatment. A method for measuring the surface of a cone, which is accurate and geometric follows from a consideration similar to that of § 649. In this manner the truth of the following statement is established.

The lateral area of a right circular cone is the common limit of the areas of inscribed and circumscribed regular pyramids, as the number of faces is indefinitely doubled.

719. Theorem. ***The lateral area of a right circular cone is equal to half the product of the circumference of the base and the slant height.*** $S=\frac{1}{2}cs=\pi rs$.



Given a right circular cone.

To prove $S=\frac{1}{2}cs=\pi rs$, where S denotes lateral area, s slant height, c circumference of base, and r radius.

Proof. Circumscribe a regular pyramid about the cone, and let S' denote its lateral area and p the perimeter of its base.

By doubling indefinitely the number of faces of the pyramid,

$S' \to S$, and $p \to c$. §§ 718, 488

Then $\frac{1}{2}ps \to \frac{1}{2}cs$. § 485 (2)

But $S'=\frac{1}{2}ps$, being variables that are always equal. § 711

$\therefore S=\frac{1}{2}cs$. § 485 (1)

Also $c=2\pi r$. § 495

$\therefore S=\pi rs$. § 111

Remark. It is to be noted that the formulas of this article apply *only* to right circular cones.

720. Theorem. *The total area T of a right circular cone is given by the formula* $T=\pi rs+\pi r^2=\pi r(s+r)$.

EXERCISES

1. Form a cone from a semicircle having a radius of 3 in. Find the lateral area of the cone. The total area.

2. Show how to cut a pattern for a tin cone that is to have the diameter of its base 4 in. and its slant height 10 in.

3. In a right circular cone show that $s=\frac{2S}{c}=\frac{S}{\pi r}$, $c=\frac{2S}{s}$, and $r=\frac{S}{\pi s}$.

4. Solve $T=\pi rs+\pi r^2$ and find $r=\frac{-\pi s+\sqrt{\pi^2 s^2+4\pi T}}{2\pi}$, and $s=\frac{T-\pi r^2}{\pi r}=\frac{T}{\pi r}-r$.

In a right circular cone, using S for lateral area, T for total area, s for slant height, r for radius, c for circumference, and h for altitude, solve the following:

5. Given $r=4$ in., $s=6$ in.; find S and T.

6. Given $c=25$ in., $s=8$ in.; find S and T.

7. Given $s=8$ ft. 6 in., $c=3$ ft. 7 in.; find S.

8. Given $s=7.2$ in., $r=5.3$ in.; find S and T.

9. Given $h=12$ in., $r=5$ in.; find S and T.

10. Given $c=30$ in., $s=8$ in.; find h and S.

11. Given $S=200$ sq. in., $r=6$ in.; find s.

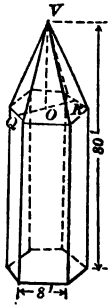
12. Given $S=256$ sq. cm., $s=16$ cm.; find r.

13. Given $T=500$ sq. cm., $r=10$ cm.; find s.

14. The circumference of the base of a conical church steeple is 35 ft. and the altitude is 73 ft. Find the lateral area. *Ans.* 1281−sq. ft.

15. Find the lateral edge and the lateral area of a regular pyramid each side of whose triangular base is 10 ft., and whose altitude is 18 ft. *Ans.* 18.90+ ft., 273.45− sq. ft.

16. A tower with a regular hexagonal base has dimensions as shown in the figure. The pitch of the pyramdial roof is $1\frac{1}{2}$, which means that $OV=1\frac{1}{2}\ QR$. Find the lateral area of the tower.



17. Cut out of heavy paper a sector of a circle, with a radius of 3 in. and the central angle 120°. Bring the edges together and paste them. Find the lateral area and the total area of the cone thus formed.

18. The cone formed by revolving an equilateral triangle about one of its altitudes has a lateral area equal to twice the area of the base.

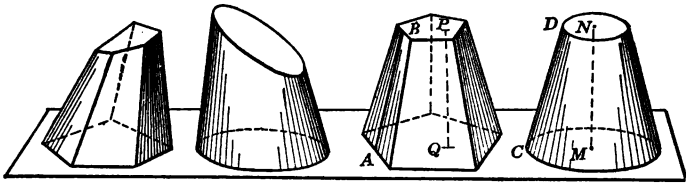
19. Find the area of the surface of the solid formed by revolving an equilateral triangle, having a side of 12 in., about one of its sides.

20. Find the area of the surface of the solid formed by revolving a right triangle, having a base of 8 in. and an altitude of 6 in., about its hypotenuse.

AREA OF FRUSTUM OF PYRAMID OR CONE

721. Truncated Pyramid, or Cone. The portion of a pyramid, or cone, included between the base and a section not parallel to the base is called a **truncated pyramid,** or **cone.**

722. Frustum. The portion of a pyramid, or cone, included between the base and a section parallel to the base is called a **frustum of the pyramid,** or **cone.**



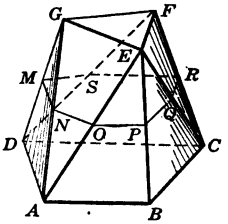
723. The base and the parallel section are called the **bases** of the frustum. The perpendicular distance between the bases is the **altitude** of the frustum.

724. The **slant height** of a frustum of a regular pyramid, or of a cone, is that portion of the slant height of the pyramid, or cone, included between the bases of the frustum.

Thus AB and CD are slant heights, and QP and MN are altitudes.

725. Midsection. If a solid has parallel bases, the section parallel to the bases and half way between them is called the **midsection.**

In the figure, NR is the midsection.

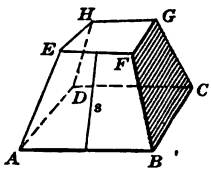


726. Prove the following facts concerning frustums:

(1) *The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.*

(2) *The lateral edges of a frustum of a regular pyramid are equal; and the slant height is the same for all the faces.*

727. Theorem. ***The lateral area of a frustum of a regular pyramid is equal to half the product of the slant height and the sum of the perimeters of the bases.*** $S=\frac{1}{2}s(P+p)$.



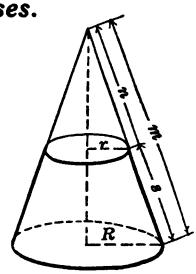
Given the frustum of a regular pyramid with bases AC and EG.

To prove that $S=\frac{1}{2}s(P+p)$, where S denotes lateral area, s slant height, and P and p the perimeter of the lower and upper base respectively.

Proof.

	Area $AF=\frac{1}{2}s(AB+EF)$.	§ 370
	Area $BG=\frac{1}{2}s(BC+FG)$.	
	etc.	
Hence	$S=\frac{1}{2}s[(AB+BC+\cdots)+(EF+FG+\cdots)]$.	§ 105
But	$AB+BC+\cdots=P$, and $EF+FG+\cdots=p$.	Why?
	$\therefore S=\frac{1}{2}s(P+p)$.	Why?

728. Theorem. ***The lateral area of a frustum of a right circular cone is equal to half the product of the slant height and the sum of the circumferences of the bases.***



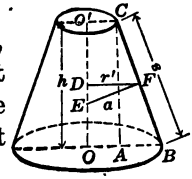
Given the frustum of a right circular cone.

To prove $S=\pi(R+r)s$, where S denotes lateral area, s slant height, and R and r the radius of the lower and upper base respectively.

Proof. Let m be the slant height of the cone and n the slant height of the part above the frustum.

Then	$S=m\pi R-n\pi r=\pi(mR-nr)$.	Why?
But	$R:r=m:n$.	§ 428
And therefore	$mr-nR=0$.	§ 398
Hence	$S=\pi(mR-nr+mr-nR)$	Why?
	$=\pi(R+r)\ (m-n)$.	Why?
But	$m-n=s$.	
	$\therefore S=\pi(R+r)s$.	Why?

729. Theorem. *The lateral area of a frustum of a right circular cone is equal to the product of the altitude and the circumference of a circle whose radius is the perpendicular at the midpoint of an element, and terminated by the axis.*



Outline of proof. Let a be the length of EF, the perpendicular at the midpoint of an element and terminated by the axis. Let r' be the radius of the midsection, and s and h the slant height and altitude respectively.

Show that $\triangle DEF \backsim \triangle ABC$.

Then $s : h = a : r'$, and $sr' = ah$.

Show that $R + r = 2r'$, where R and r are the radii of the bases.

Then, substituting in the formula of § 728, $S = 2\pi ah$.

EXERCISES

1. Prove the theorem of § 728 by considering the lateral area of a frustum of a right circular cone as the limit of the area of a frustum of a pyramid.

2. Prove the theorem of § 727 by using a method similar to that of § 728 for the frustum of a cone.

3. Show that the lateral area of a frustum of a right circular cone is equal to the slant height times the perimeter of the midsection.

4. Find the lateral area of the frustum of a regular quadrangular pyramid, if an edge of the lower base is 16 in., an edge of the upper base 12 in., and the slant height 18 in.

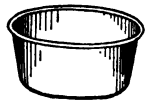
5. Find the total area of the frustum of a right circular cone, if the radii of the bases are 6 ft. and 8 ft. respectively, and the slant height is 7 ft.

6. A portion of the roof of a tower is a frustum of a right circular cone. The radii of the bases are 10 ft. and 6 ft. respectively, and the altitude is 8 ft. Find the number of square feet in this part of the roof.

7. The altitude of a right circular cone is h. How far from the vertex must a plane be passed parallel to the base so that the lateral area of the cone cut off shall equal the lateral area of the frustum. *Ans.* $\frac{1}{2}h\sqrt{2}$.

8. The diameter of the bottom of a pail is 10 in. and that of the top 12 in. Find the number of square inches of tin in the pail if the slant height is 11 in. *Ans.* 458.67.

9. Determine the diameter of the blank to make a pressed basin of the form shown in the figure. The depth is 3 in., the bottom has a diameter of 6 in., the top an inside diameter of 7 in., and the rim is $\frac{1}{2}$ in. wide.



Ans. 11.4+ in.

SUGGESTION. Consider the rim as a ring between two concentric circles. The blank must have an area equal to the total area of the basin.

10. A pie tin has a diameter of $7\frac{1}{3}$ in. at the bottom and 9 in. at the top inside. It is 1 in. deep and has a flange $\frac{3}{8}$ in. wide around the edge. Find the diameter of the blank required to make the tin.

11. Find the area generated by revolving a square a inches on a side about a diagonal.

12. A tower whose cross section is a regular octagon 6 ft. on a side has a roof that is full pitch. Find the area of the roof.

By full pitch is meant that the vertex of the roof is the same distance above the plates as the width of the tower from face to face.

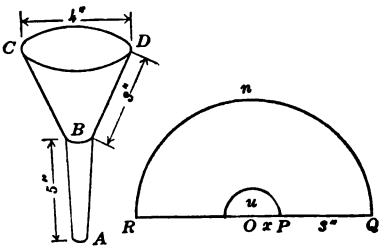
13. The conical roof over a water tank is half pitch. Find its area if the diameter of the tank is 12 ft. and the roof projects $1\frac{1}{2}$ ft. at the eaves.

Ans. 221.5— sq. ft.

By half pitch is meant that the vertex of the roof is one-half the diameter of the tower above the top of the tower.

14. Find the total area of the solid generated by revolving an isosceles trapezoid about a line connecting the middle points of its bases, if the bases are 8 in. and 12 in. respectively, and the angles at one base are each 60°.

15. Determine how to draw a pattern for the upper part of the funnel with dimensions as shown, the diameter at A being $\frac{1}{2}$ in. and at B, 1 in. The allowance for the seam can be added.



SUGGESTION. The length $OP = x$ may be determined as follows: $x : x+3 = 1 : 4$. $\therefore x = 1$.

The radius to use is therefore 4 in.

To determine the central angle u, $\frac{\overset{\frown}{QnR}}{8\pi} = \frac{u}{360°}$.

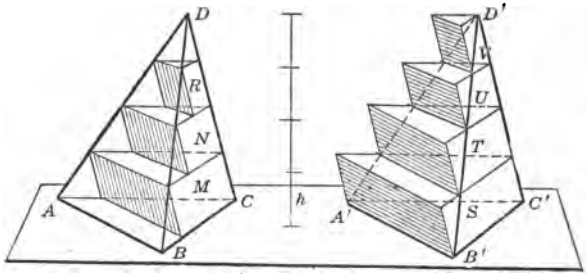
But $\overset{\frown}{QnR} = 4\pi$. $\therefore u = 180°$.

16. Determine how to draw a pattern for the lower part of the funnel shown in the figure of the previous exercise.

VOLUMES OF PYRAMIDS AND CONES

730. Inscribed Prisms. If the altitude of a triangular pyramid is divided into equal parts by a series of planes parallel to the base, the prisms within the pyramid and having the sections formed by the parallel planes as bases, and each having one of the segments of one of the lateral edges of the pyramid as one of its lateral edges, are called a **set of inscribed prisms.**

In the figure M, N, and R are a set of inscribed prisms, each being entirely within the pyramid.



731. Circumscribed Prisms. In a similar manner, the prisms having the base of the pyramid and the sections as bases are called a **set of circumscribed prisms.**

In the figure, S, T, U, and V are a set of circumscribed prisms, each being partly outside of the pyramid.

732. The two sets of such prisms, formed by using the same lateral edge and the same parallel planes, are called **corresponding sets of inscribed and circumscribed prisms.**

733. The following statement may be considered as evident from the preceding:

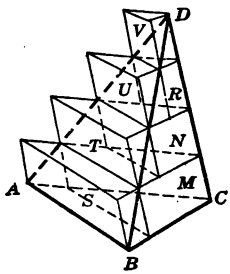
The volume of a triangular pyramid is definite, and is greater than that of any set of inscribed prisms, and less than that of any set of circumscribed prisms.

734. Theorem. *The difference in volume between a set of inscribed prisms and the corresponding set of circumscribed prisms is the volume of the prism upon the base of the pyramid.*

735. Theorem. *The volume of a triangular pyramid is the common limit of the volumes of the set of inscribed prisms and the corresponding set of circumscribed prisms, as the number of these prisms is increased indefinitely.*

Given the triangular pyramid *D-ABC*, a set of inscribed prisms, and corresponding set of circumscribed prisms.

To prove that volume of *D-ABC* is the common limit of these two corresponding sets of prisms.



Outline of proof. The volume of the triangular pyramid is greater than the sum of the inscribed prisms and less than the sum of the circumscribed prisms. § 733

Therefore, the volume of the pyramid differs from either by an amount less than the volume of the prism upon the base of the pyramid. § Why?

But the volume of this prism can be made as small as desired, that is, it has zero for a limit. § 485 (2)

Hence, the volume of the set of inscribed prisms and the set of circumscribed prisms, each have the volume of the triangular pyramid for a limit. § 483

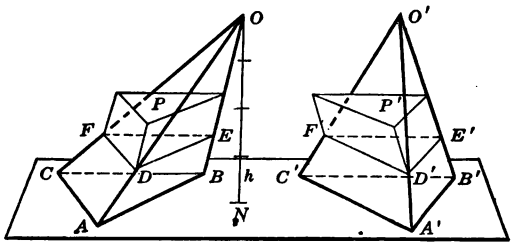
EXERCISES

1. Corresponding sets of inscribed and circumscribed prisms are formed in a triangular pyramid whose base has an area of 25 sq. in. Compute the series of differences between the corresponding sets when their altitudes are successively 0.1 in., 0.01 in., 0.001 in., and 0.0001 in.

2. The base of a pyramid is an equilateral triangle 3 in. on a side and its altitude is 7 in. Find the difference between the corresponding sets of inscribed and circumscribed prisms when the altitude of each prism is $\frac{1}{1000}$ of the altitude of the pyramid.

3. Could a quadrangular pyramid be used instead of a triangular pyramid in the discussion of §§ 730-733?

736. Theorem. *The volumes of two triangular pyramids, having equivalent bases and equal altitudes, are equivalent.*



Given the triangular pyramids $O\text{-}ABC$ and $O'\text{-}A'B'C'$, having equivalent bases and equal altitudes; and, for convenience, having their bases in the same plane.

To prove that $V = V'$, where V denotes the volume of $O\text{-}ABC$ and V' the volume of $O'\text{-}A'B'C'$.

Proof. Let the common altitude ON be divided into any number of equal parts, and pass planes through the points of division parallel to the plane of the bases. Construct the set of circumscribed prisms for each pyramid upon the sections formed by these planes.

Then the sections of the pyramids made by each plane are equivalent. § 710

Let DEF and $D'E'F'$ be the sections made by any one of these planes, and P and P' the corresponding prisms.

Then volume of P = volume of P'. § 686

Let S denote the volume of the sum of the prisms in the set of which P is one; and S' that of which P' is one.

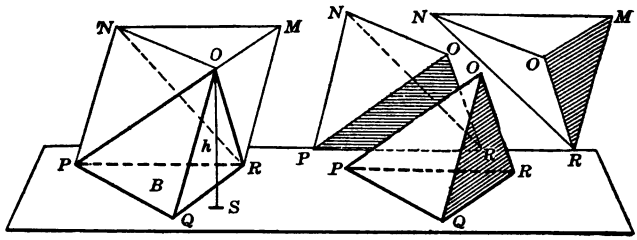
Then $S = S'$. § 105

But $S \rightarrow V$, and $S' \rightarrow V'$. § 735

And S and S' are variables that are equal for all their successive values as the number of the divisions of ON is increased indefinitely.

$\therefore V = V'$. § 485 (1)

737. Theorem. ***The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*** $V = \frac{1}{3}Bh$.



Given the triangular pyramid $O\text{-}PQR$.

To prove that $V = \frac{1}{3}Bh$, where V denotes the volume of the pyramid, B the area of its base, and h its altitude.

Proof. Upon the base PQR construct a triangular prism NR of altitude h, and having its lateral edges parallel to OQ.

The prism NR is divided into three triangular pyramids by the sections OPR and ONR.

Pyramid $R\text{-}NOM$ = pyramid $O\text{-}PQR$. § 736

Pyramid $R\text{-}OPQ$ = pyramid $R\text{-}ONP$. § 736

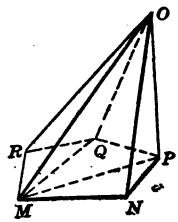
But pyramid $R\text{-}OPQ$ is the same as pyramid $O\text{-}PQR$.

Therefore the three triangular pyramids are equal, and $O\text{-}PQR$ is one-third the volume of prism NR.

But volume of prism $NR = Bh$. § 684

$\therefore V = \frac{1}{3}Bh$. Why?

738. Theorem. *The volume of any pyramid is equal to one-third the product of its base and altitude.* $V = \frac{1}{3}Bh$.

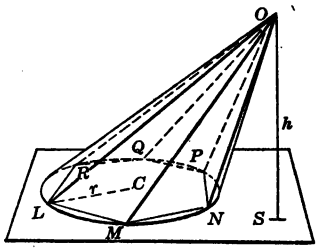


Suggestion. From one vertex of the base draw all the diagonals of the base. Pass planes through the vertex of the pyramid and each of these diagonals. All the pyramids have the same altitude. Apply § 737 to each triangular pyramid formed and add the results.

739. By a like treatment to that referred to in § 650, the truth of the following statement may be established:

The volume of a circular cone is the common limit of the volumes of inscribed and circumscribed pyramids with regular polygons for bases, as the number of faces is indefinitely doubled.

740. Theorem. ***The volume of a circular cone is equal to one-third the product of its base and altitude.*** $V=\frac{1}{3}Bh=\frac{1}{3}\pi r^2h$.



Given a circular cone.

To prove $V=\frac{1}{3}Bh=\frac{1}{3}\pi r^2h$, where V denotes volume, B area of base, h altitude, and r radius.

Proof. Inscribe a pyramid with a regular polygon for base, and let V' denote its volume and B' the area of its base.

By doubling indefinitely the number of faces of the pyramid,

$V'\to V$, and $B'\to B$. § 739, § 489

$\frac{1}{3}B'h\to\frac{1}{3}Bh$. § 485 (2)

But $V'=\frac{1}{3}B'h$, being variables that are always equal. § 738

$\therefore V=\frac{1}{3}Bh$. § 485 (1)

Also $B=\pi r^2$. § 498

$\therefore V=\frac{1}{3}\pi r^2h$. § 111

Remark. It is to be noted that the formulas of this article apply to *any* circular cone. Compare with the remark of § 719.

EXERCISES

1. Show that for any pyramid or circular cone $B=\frac{3V}{h}$, and $h=\frac{3V}{B}$.

2. Find the volume of a square pyramid whose base is 16 in. on a side, and whose altitude is 10 in.

3. Find the volume of a pyramid whose base is a regular hexagon 8 in. on a side, and whose altitude is 14 in.

4. Find the volume of a pyramid whose base is a square $3\sqrt{2}$ in. on a side, and whose altitude is $6\frac{3}{4}$ in.

5. The base edges of a triangular pyramid are 6 in., 8 in., and 10 in. Find its volume if the altitude is 15 in.

6. Find the volume of a circular cone whose radius is 5 in. and altitude 10 in.

7. Find the volume of a circular cone whose altitude is 70 ft. and the circumference of whose base is 31 ft.

8. Derive a formula for finding the radius of a circular cone in terms of the volume and altitude.

9. Derive a formula for finding the altitude of a circular cone in terms of the volume and radius.

10. A right circular cone has a slant height s and a radius r; find its volume V. *Ans.* $V=\frac{1}{3}\pi r^2\sqrt{s^2-r^2}$.

11. The Pyramid of Cheops has a square base 720 ft. on a side, and an altitude of 480 ft. Find the number of cubic yards in it.

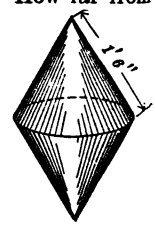
12. What is the locus of the vertices of all pyramids having the same base and equal volumes?

13. Find the total area and the volume of the solid generated by revolving an equilateral traingle of side a about one side. Find the values if $a=2$ in. *Ans.* Area $=a^2\pi\sqrt{3}$; 21.765 sq. in.
Volume $=\frac{1}{4}a^3\pi$; 6.283 cu. in.

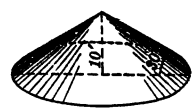
14. Find the total area and the volume of the solid generated by revolving a parallelogram with sides 22 in. and 16 in., and larger angle 120°. about one of the longer sides. *Ans.* 3308.3− sq. in.; 13270+ cu. in.

15. A pyramid whose base is a square 4 ft. on a side, and whose altitude is 12 ft. is bisected by a plane parallel to the base. How far from the vertex is the plane?

16. A circular sheet of copper 3 ft. in diameter is cut in half, and each half formed into a cone. These are placed base to base as shown in the figure. This is to be used as a float. Find the number of pounds it will support if the sheet copper weighs 0.92 lb. per square foot and water weighs 62.5 lb. per cubic foot.

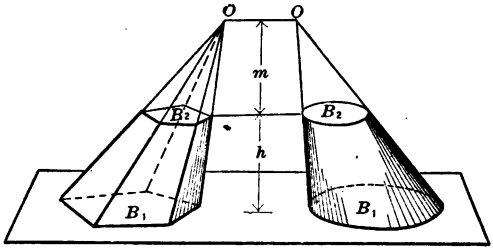


17. Hard coal dumped in a pile lies at an angle of 30° with the horizontal. Estimate the number of tons in a pile in the shape of a right circular cone having an altitude of 10 ft. Large egg size weighs 38 lb. per cubic foot. *Ans.* About 60 tons.



18. Show how to construct a triangular pyramid equivalent to any given pyramid having a polygon as base

741. Theorem. ***The volume V of a frustum of a pyramid, or circular cone, of bases B_1 and B_2 and altitude h, is given by the formula*** $V=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$.



Given the frustum of a pyramid, or of a circular cone.

To prove $V=\frac{1}{3}h(B_1+B_2+\sqrt{B_1B_2})$, where V denotes volume, h altitude, and B_1 and B_2 the areas of the lower base and upper base, respectively.

Proof. Complete the pyramid, or cone, and let m denote the altitude of the portion above the frustum in each case, and V_2 the volume of that portion. Also let V_1 denote the volume of the entire pyramid, or cone.

Then
$$V=V_1-V_2.$$
$$=\tfrac{1}{3}B_1(h+m)-\tfrac{1}{3}B_2m. \qquad \text{§ 740}$$
$$=\tfrac{1}{3}[B_1h+(B_1-B_2)m].$$

It remains to eliminate m. To do this one more equation is needed. This is given by

$$\frac{B_1}{B_2}=\frac{(h+m)^2}{m^2}. \qquad \text{§ 709}$$

$$\frac{\sqrt{B_1}}{\sqrt{B_2}}=\frac{h+m}{m}.$$

Solving for m,
$$m=\frac{h\sqrt{B_2}}{\sqrt{B_1}-\sqrt{B_2}}.$$

Substituting this for m,
$$V=\tfrac{1}{3}h(B_1+B_2+\sqrt{B_1B_2}).$$

742. Theorem. *The volume V of a frustum of a circular cone having bases with diameters d_1 and d_2, and radii r_1 and r_2, and an altitude h, is given by the formulas:*

$$(1)\quad V=\tfrac{1}{3}\pi h(r_1^2+r_2^2+r_1r_2),$$

$$(2)\quad V=\tfrac{1}{12}\pi h(d_1^2+d_2^2+d_1d_2).$$

Substitute πr_1^2 and πr_2^2 for B_1 and B_2 respectively of § 741 to find (1). Use $\frac{1}{4}\pi d_1^2$ and $\frac{1}{4}\pi d_2^2$ for (2).

EXERCISES

1. Find the volume of a frustum of a pyramid with square bases having edges of 9 ft. and 4 ft. respectively, and altitude 12 ft.

2. The radii of the bases of a frustum of a right circular cone are respectively 10 cm. and 12 cm. Find its volume if the altitude is 16 cm. Find its lateral area.

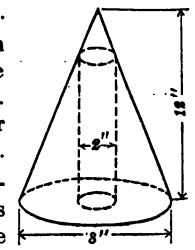
3. Find the volume of a frustum of a regular hexagonal pyramid the bases being respectively 8 in. and 6 in. on an edge, and the altitude 10 in.

4. The frustum of a cone has an altitude of 8 in. and the radii of its bases are 7 in. and 4 in. respectively. Find its volume. Does the frustum have to be cut from a right cone in order that its volume can be found?

5. Derive a formula for finding the altitude of a frustum of a circular cone in terms of the volume and the radii of the bases.

6. The diameter of the top of a water pail is 12 in., the diameter of the bottom 10 in., and the altitude 10½ in. How many quarts will the pail hold? One quart is 57.75 cu. in. *Ans.* 17.33−.

7. Find the weight of an iron casting in the form of a right circular cone of diameter 8 in. and altitude 12 in., with a circular hole having a diameter of 2 in. through the center. (Cast iron weighs 0.26 lb. per cubic inch.) *Ans.* 44.1 lb.



8. A tank is made in the form of an inverted frustum of a cone. The slant height is 14 ft. and makes an angle of 60° with the horizontal, and the lower base is a circle 20 ft. in diameter. Find the volume of the tank in barrels. Use 1 bbl. = 4.211 cu. ft. *Ans.* 1685.4 bbl.

9. A tank is in the form of an inverted cone with its vertex angle 60° and its axis vertical. Find the depth of water in the cone if it has been running in for 10 minutes at the rate of 8 cu. ft. per minute.

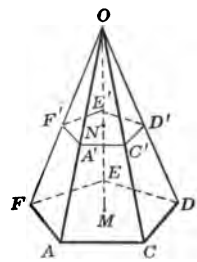
SIMILAR PYRAMIDS AND CONES

743. Similar Pyramids. Pyramids that are formed from the same pyramidal surface by parallel planes are **similar.**

744. Similar Cones. Cones that are formed from the same conical surface by parallel planes are **similar.**

745. Theorem. *The lateral areas, or the total areas, of two similar regular pyramids, or right circular cones, are to each other as the squares of two corresponding lines, where the corresponding lines may be edges, elements, altitudes, slant heights or radii.*

One case only will be proved. The other cases are left as exercises.



Given two similar regular pyramids $O\text{-}ACDEF$ and $O\text{-}A'C'D'E'F'$.

To prove $\frac{S}{S'}=\frac{h^2}{h'^2}$, and $\frac{T}{T'}=\frac{h^2}{h'^2}$, where S and S' denote the lateral areas of $O\text{-}ACDEF$ and $O\text{-}A'C'D'E'F'$ respectively, h and h' altitudes, and T and T' total areas.

Proof. If p and p' are the perimeters of the respective bases, and s and s' the slant heights, $S=\frac{1}{2}ps$, and $S'=\frac{1}{2}p's'$. § 711

Then $$\frac{S}{S'}=\frac{\frac{1}{2}ps}{\frac{1}{2}p's'}=\frac{p}{p'}\cdot\frac{s}{s'}.$$ Why?

$$\therefore \frac{S}{S'}=\frac{h^2}{h'^2} \text{ since } \frac{p}{p'}=\frac{h}{h'}, \text{ and } \frac{s}{s'}=\frac{h}{h'}.$$ Why?

Further, if B and B' are the areas of the respective bases,

$$\frac{B}{B'}=\frac{h^2}{h'^2} \text{ and so } \frac{S}{S'}=\frac{B}{B'}, \text{ and } \frac{S}{B}=\frac{S'}{B'}.$$ Why?

Then $$\frac{S+B}{B}=\frac{S'+B'}{B'}, \text{ and } \frac{S+B}{S'+B'}=\frac{B}{B'}.$$ §§ 406, 405

$$\therefore \frac{T}{T'}=\frac{h^2}{h'^2} \text{ since } S+B=T, \text{ and } S'+B'=T'.$$ Why?

746. Theorem. *The volumes of two similar pyramids, or circular cones, are to each other as the cubes of two corresponding lines; where the corresponding lines may be edges, elements, altitudes, slant heights, or radii.*

Using V and V' for the respective volumes, B and B' for bases, and h and h' for altitudes,

$$\frac{V}{V'}=\frac{\frac{1}{3}Bh}{\frac{1}{3}B'h'}=\frac{B}{B'}\cdot\frac{h}{h'}.$$

But $\frac{B}{B'}=\frac{h^2}{h'^2}$. Why? $\therefore \frac{V}{V'}=\frac{h^3}{h'^3}$. Why?

747. Definition. The **vertex angle** of a right circular cone is twice the angle between an element and the axis.

748. Theorem. *All right circular cones with equal vertex angles are similar.*

EXERCISES

1. Two cones generated by similar right triangles revolving about corresponding sides are similar.

2. Two cones are generated by revolving two similar right triangles about corresponding sides of length 7 in. and 10 in. respectively. Find the ratio of the lateral areas of the cones. The ratio of the total areas. The ratio of the volumes.

3. The total area of a right circular cone is 300 sq. in. and its altitude is 8 in. Find the total area of the cone cut off by a plane parallel to the base and 6 in. from the vertex.

4. How far from the vertex of a right circular cone, or a regular pyramid, of altitude h must a plane be passed parallel to the base so that the lateral area of the part cut off shall be one-half that of the original?

5. From a given pyramid, cut off a frustum whose volume shall be $\frac{26}{27}$ that of the given pyramid. Must the pyramid be regular?

6. The volumes of two pyramids are 343 cu. in. and 1000 cu. in. respectively. The lateral area of the smaller is 100 sq. in. Find the lateral area of the larger.

7. The lateral areas of two similar right circular cones are in the ratio of 9 : 64. Find the volume of the smaller if the volume of the larger is 1024 cu. in.

QUESTIONS

1. What is a pyramidal surface? A conical surface? Is the directrix necessarily closed?

2. What are the formulas or rules for finding the areas of pyramids, cones, and frustums of pyramids and cones?

3. What are the formulas or rules for finding the volumes of each of these solids?

4. Can you find the area of an oblique circular cone? Of an oblique pyramid? Can you find the volume of each of these?

5. If the altitude of a cone is divided into 10 equal parts by planes passed parallel to the base, how does the area of each section of the cone formed by these planes compare with the base of the cone?

6. Trace the steps in finding the volume of a triangular pyramid. Of any pyramid. Of a cone.

7. What is the relation between two parallel sections of a pyramid or cone?

8. State theorems concerning the sections of a cone. Of a pyramid.

9. What effect does it have upon the volume of a pyramid or cone if the area of the base is doubled? If the altitude is doubled? If both base and altitude are doubled?

10. What effect does it have upon the lateral area of a right circular cone if the circumference of the base is doubled? If the area of the base is doubled?

11. Point out forms of this chapter that occur in nature. That occur in the arts.

GENERAL EXERCISES

1. Given the following formulas for the right circular cone: (1) $V = \frac{1}{3}\pi r^2 h$, (2) $S = \pi r s$, (3) $S = \pi r\sqrt{h^2+r^2}$, (4) $T = A + \pi r\sqrt{h^2+r^2}$, (5) $T = \pi r^2 + \pi r\sqrt{h^2+r^2}$; solve each for each letter appearing.

2. In a right circular cone, $A = 50\pi$, and $V = 200\pi$. Find r, h, and s.

3. If R and r are the radii of the two bases of a frustum of a cone, show that $R + r = \sqrt{\frac{3V}{\pi h} + Rr}$.

4. Find the volume and the total area of a right circular cone whose radius of base is 6 in. and altitude is 5.3 in. *Ans.* 199.8 cu. in.; 263.9 sq. in.

5. Find the weight of a tapered brick smoke stack 175 ft. high in the form of a frustum of a right circular cone enclosing a cylinder. The inner diameter is 10 ft., the wall is 4 ft. thick at the base and 1 ft. 6 in. at the top. One cubic foot of brick weighs 112 pounds. *Ans.* 1095.5 tons.

6. Find the lateral edge, lateral area, and volume of a regular pyramid each side of whose triangular base is 5 ft., and whose altitude is 9 ft. *Ans.* 9.45 ft.; 68.36 sq. ft.; 32.5 cu. ft.

7. A bin in a warehouse is 12 ft. square. A hopper is constructed on the base, which has a slope of 1 : 1. The distance from the vertex of the hopper to the top of the bin is 18 ft. Find the capacity of the bin if 1 bushel equals $\frac{5}{4}$ cu. ft.

8. The largest possible cylinder of diameter 6 in. is cut from a right circular cone having a diameter and altitude of 10 in. and 26 in. respectively. Find the volume of the cylinder.

9. Find the area and volume of the solid generated by revolving an isosceles triangle of base 10 in. and equal sides 16 in., about its base. Find the area and the volume of the solid when this triangle is revolved about one of its equal sides.

10. In the frustum of a pyramid whose base is 50 sq. ft. and altitude 6 ft., the basal edge is to the corresponding top edge as 5 to 3. Find the volume of the frustum. *Ans.* 196 cu. ft.

11. The base of a pyramid is a rectangle 10 in. by 8 in. Find the volume if each lateral edge makes with the base an angle of 45°.

12. The total surface of a regular quadrangular pyramid is T, and its eight edges are equal. Find the length of an edge.

13. Find the volume of a cone of revolution whose slant height is equal to the diameter of its base, and whose total area is T.

14. Show how to cut a pattern for the frustum of a right circular cone, if the upper and lower bases have diameters of 4 in. and 6 in., respectively, and the altitude is 8 in.

15. A right circular cone whose altitude is 8 in. and radius 6 in. rolls on a floor without slipping, making one complete revolution. What is the shape of the surface covered? Find its area.

16. A pyramid whose altitude is 12 in. weighs 30 pounds. At what distance from the vertex must it be cut by a plane parallel to its base so that the frustum cut off shall weigh 20 pounds?

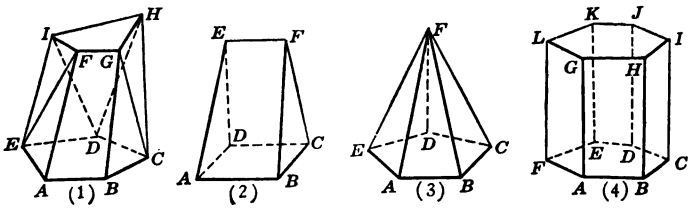
17. A cube is cut by a plane passed through the other extremities of the three edges meeting at a vertex. What part of the volume of the cube is thus cut off?

CHAPTER IX

PRISMATOIDS AND POLYEDRONS

PRISMATOIDS

749. A polyedron all of whose vertices lie in two parallel planes is called a **prismatoid.** In either plane, the vertices may lie in any polygon, as Fig. (1); or, in one plane, the vertices may lie in a line, as in Fig. (2); or there may be a point only, as in a pyramid. The faces whose vertices are in the



parallel planes are evidently triangles, or quadrilaterals of types that have at least two parallel sides. Why?

750. The **altitude** of a prismatoid is the distance between the two parallel planes.

EXERCISES

1. Show that a prism is a prismatoid whose bases are congruent polygons.

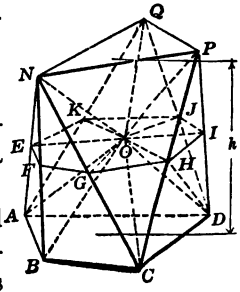
2. Show that a frustum of a pyramid is a prismatoid whose bases are similar polygons.

3. Draw or describe a prismatoid that has a face that is a trapezoid. A parallelogram. A rectangle.

4. Compute the area of the midsection of a pyramid whose base is a regular hexagon having a side of 6 in.

5. Compute the areas of the midsection of the prismatoid in Fig. (2) if $ABCD$ is a rectangle having $AB = 18$ in. and $BC = 11$ in., and $EF = 12$ in.

751. Theorem. *If B_1 and B_2 are the areas of the two bases, M the area of the midsection, and h the altitude, then the volume V of a prismatoid is given by the formula* $V=\frac{1}{6}h(B_1+B_2+4M)$.

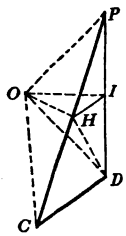


Given a prismatoid of volume V, altitude h, bases B_1 and B_2, and midsection M.

To prove $V=\frac{1}{6}h(B_1+B_2+4M)$.

Proof. If any lateral face is not a triangle divide the face into triangles by a diagonal.

Take any point O in the midsection and connect it with the vertices of the prismatoid and the midsection. These lines form triangles with the edges of the prismatoid, which separate the prismatoid into pyramids that have the bases and faces of the prismatoid as bases.



(1) Pyramid $O\text{-}ABCD=\frac{1}{6}hB_1$. § 737

(2) Pyramid $O\text{-}NPQ=\frac{1}{6}hB_2$.

Each of the pyramids like $O\text{-}CDP$, which may be called a lateral pyramid, can have its volume expressed in terms of the part of the midsection common to it. To prove this consider $O\text{-}CDP$ apart from the prismatoid, and draw HD.

Altitude of each of pyramids $P\text{-}OHI$ and $D\text{-}OHI$ is $\frac{1}{2}h$.

Pyramid $P\text{-}OHI=\frac{1}{6}h$ times area OHI. § 737

Pyramid $D\text{-}OHI=\frac{1}{6}h$ times area OHI. Why?

Pyramid $O\text{-}CDH=2$ times pyramid $O\text{-}HDI$. Why?

Then pyramid $O\text{-}CDH=\frac{2}{6}h$ times area OHI.

Hence pyramid $O\text{-}CDP=\frac{4}{6}h$ times area OHI. Why?

Similarly each lateral pyramid is equal to $\frac{4}{6}h$ times the area of its part of the midsection.

(3) Therefore the sum of the lateral pyramids $=\frac{4}{6}hM=\frac{1}{6}h\cdot 4M$. Why?

Adding (1), (2), and (3),

$$V=\tfrac{1}{6}h(B_1+B_2+4M).$$

752. Other Applications of the Prismatoid Formula. The formula for the volume of a prismatoid can be applied also to various other solids such as cylinders, cones, spheres, combinations of these, and various other forms that cannot well be described here.

The prismatoid formula, because of its wide application, is used often by engineers and others in determining the volumes of embankments, abutments, and other irregular solids of various kinds.

EXERCISES

1. Derive the following formulas from the formula for the volume of a prismatoid:

(1) Prism and cylinder, $V = Bh$.

(2) Pyramid and cone, $V = \frac{1}{3}Bh$.

(3) Frustum of pyramid or cone, $V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.

Derivation for frustum.

Given $V = \frac{1}{6}h(B_1 + B_2 + 4M)$.

To derive $V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.

It is then necessary to express M in terms of B_1 and B_2.

Using the notation of § 741, $\frac{B_1}{M} = \frac{(h+m)^2}{(\frac{1}{2}h+m)^2}$, and $\frac{B_2}{M} = \frac{m^2}{(\frac{1}{2}h+m)^2}$. § 709

Or $\frac{\sqrt{B_1}}{\sqrt{M}} = \frac{h+m}{\frac{1}{2}h+m}$, and $\frac{\sqrt{B_2}}{\sqrt{M}} = \frac{m}{\frac{1}{2}h+m}$.

Solving each of these for m, $m = \frac{h(\sqrt{M} - \frac{1}{2}\sqrt{B_1})}{\sqrt{B_1} - \sqrt{M}}$ from the first;

and $m = \frac{\frac{1}{2}h\sqrt{B_2}}{\sqrt{M} - \sqrt{B_2}}$ from the second.

Then $\frac{h(\sqrt{M} - \frac{1}{2}\sqrt{B_1})}{\sqrt{B_1} - \sqrt{M}} = \frac{\frac{1}{2}h\sqrt{B_2}}{\sqrt{M} - \sqrt{B_2}}$.

Dividing by $\frac{1}{2}h$, $\frac{2\sqrt{M} - \sqrt{B_1}}{\sqrt{B_1} - \sqrt{M}} = \frac{\sqrt{B_2}}{\sqrt{M} - \sqrt{B_2}}$.

Then $\frac{\sqrt{M}}{2\sqrt{M} - \sqrt{B_1}} = \frac{\sqrt{M}}{\sqrt{B_2}}$, and $2\sqrt{M} = \sqrt{B_1} + \sqrt{B_2}$. § 406

Or $4M = B_1 + B_2 + 2\sqrt{B_1 B_2}$.

$\therefore V = \frac{1}{6}h(2B_1 + 2B_2 + 2\sqrt{B_1 B_2}) = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.

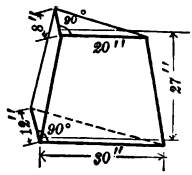
2. Show that, in general, the midsection of a prismatoid has as many sides as the two bases combined. When will this not be so?

3. Find the volume of the frustum of a square pyramid, the lower base being 8 ft. square, the upper base 6 ft. square, and the altitude 12 ft. Solve both by the frustum formula and the prismatoid formula, and compare.

4. Both bases of a prismatoid of altitude h are squares, and the lateral faces isosceles triangles. The sides of the upper base are parallel to the diagonals of the lower base and half the length of these diagonals. If b is a side of the lower base, find the volume. *Ans.* $\frac{5}{6}b^2h$.

5. Find the volume of the prismatoid with dimensions as shown in the figure. The bases are right triangles having their corresponding sides parallel.

Ans. 3420 cu. in.

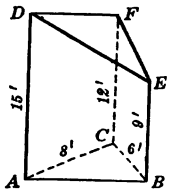


6. Use the prismatoid rule to find the volume of a frustum of a pyramid whose bases are regular hexagons 10 in. and 6 in. on a side respectively, and whose altitude is 18 in.

7. Find the volume of the frustum of the preceding exercise by § 741 and compare the result with that of the preceding.

8. The volume of a truncated triangular prism is readily determined by the prismatoid formula. To do this consider one face and the opposite edges as the bases.

Find the volume of the truncated triangular prism shown in the figure. The base ABC, which has a right angle at C, is a right section.



9. Show that the volume of any truncated triangular prism is equal to the product of the area of a right section and one-third the sum of the three edges.

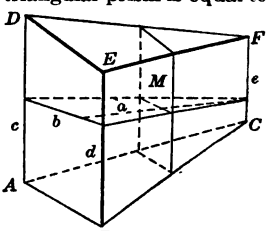
SUGGESTION. Let the edges be c, d, and e, and the altitude and base of a right section a and b respectively.

Consider the prism as a prismatoid with bases $ABED$ and CF, and midsection M.

Then $V=\frac{1}{6}a\,[\frac{1}{2}b(c+d)+0+4M]$. § 751

But $M=\frac{1}{2}b\cdot\frac{1}{2}[\frac{1}{2}(c+e)+\frac{1}{2}(d+e)]$.

$\therefore V=\frac{1}{3}ab\cdot\frac{1}{3}(c+d+e)=$ area of right section $\times\frac{1}{3}(c+d+e)$.



10. A truncated right triangular prism has for base an isosceles triangle whose sides are 10 in., 10 in., and 8 in. Find its volume if the three lateral edges are 5 in., 7 in., and 11 in. respectively.

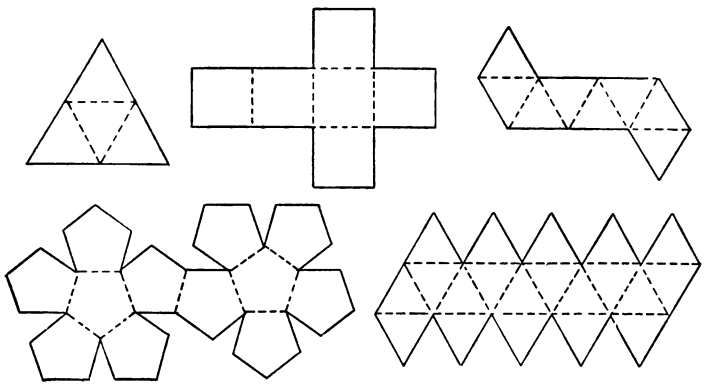
POLYEDRONS

753. Regular Polyedrons. A polyedron whose faces are congruent regular polygons and whose polyedral angles are congruent is called a **regular polyedron.**

A polyedron of four faces is a **tetraedron;** one of six faces, a **hexaedron;** one of eight faces, an **octaedron;** one of twelve faces, a **dodecaedron;** and one of twenty faces, an **icosaedron.**

It is instructive to construct the regular polyedrons as shown in the following figures. Draw on cardboard the diagrams shown below, cut along the full lines, and fold along the dotted lines. The edges that meet may be fastened with gummed paper. Durable models may be constructed from tin and soldered at the edges.

There are many interesting facts connected with the five regular polyedrons. The so-called fourteenth, fifteenth, and sixteenth books of Euclid give many theorems and problems concerning these solids.



754. Theorem. *There can be no more than five regular polyedrons.*

The proof of this theorem depends upon the facts:

(a) At least three planes must meet to form a polyedral angle. (§ 603).

(b) The sum of the face angles of a polyedral angle is less than 360°. (§ 609).

(c) The faces of a regular polyedron are congruent regular polygons.

(1) *Faces equilateral triangles.*

A polyedral angle can be formed of three, four, or five equilateral triangles, and no more. Why?

Therefore no more than three regular polyedrons can be formed having equilateral triangles as faces.

(2) *Faces squares.*

A polyedral angle can be formed of three squares, and no more. Why?

Therefore no more than one regular polyedron can be formed having squares for faces.

(3) *Faces regular pentagons.*

A polyedral angle can be formed of three regular pentagons, and no more. Why?

Therefore no more than one regular polyedron can be formed having regular pentagons for faces.

Regular polygons with a greater number of sides than five cannot be used to form polyedral angles because the sum of three or more angles of such polygons is equal to or greater than 360°.

Hence there can be no more than five regular polyedrons.

755. Relation between the Number of Faces, Edges, and Vertices of a Regular Polyedron. It is interesting to note the relation between the number of faces, edges, and vertices of a regular polyedron. Complete the following table, and compare with the table on page 67.

Name	Number of Faces	Form of Faces	Number of Edges	Number of Vertices	Number of Faces at Each Vertex	Sum of Face Angles
Tetraedron	4	Equi. △	6	4	3	720°
Hexaedron						
Octaedron						
Dodecaedron						
Icosaedron						

756. Euler's Theorem. This theorem is stated because of its interest. The proof is too difficult to be given here. As an exercise it may be tested for the regular polyedrons. Use the preceding table.

Theorem. *In any polyedron the number of edges plus two is equal to the sum of the number of faces and the number of vertices.*

NOTE. Leonhard Euler was born in Switzerland in 1707 and died in Petrograd in 1783. He was one of the greatest physicists, astronomers, and mathematicians of the eighteenth century.

757. Theorem. *The sum of the face angles of any polyedron is equal to 360° multiplied by two less than the number of vertices.*

The proof of this theorem is based upon the preceding theorem. Test it for the regular polyedrons.

EXERCISES

1. Find the area of the surface of a regular tetraedron 3 in. on an edge. Of a regular octaedron 4 in. on an edge. Of a regular icosaedron 6 in. on an edge. *Ans.* 15.588+ sq. in.; 55.426− sq. in.; 311.769+ sq. in.

2. Find the area of the surface of a regular dodecaedron 2 in. on an edge. *Ans.* 82.583− sq. in.

3. If the edge of a regular tetraedron is e, show that its volume is given by the formula $V=\frac{1}{12}e^3\sqrt{2}$.

SOLUTION. $V=\frac{1}{3}ON\times$area of $\triangle ABC$.

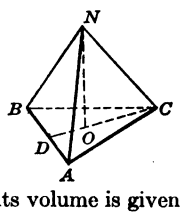
$DC=\frac{1}{2}e\sqrt{3}$. See p. 166, Ex. 7 (3).

Area of $\triangle ABC=\frac{1}{2}AB\times DC=\frac{1}{4}e^2\sqrt{3}$.

$OC=\frac{2}{3}DC=\frac{1}{3}e\sqrt{3}$. § 211.

$ON=\sqrt{\overline{CN}^2-\overline{OC}^2}=\sqrt{e^2-(\frac{1}{3}e\sqrt{3})^2}=e\sqrt{\frac{2}{3}}$.

$\therefore V=\frac{1}{3}e\sqrt{\frac{2}{3}}\times\frac{1}{4}e^2\sqrt{3}=\frac{1}{12}e^3\sqrt{2}$.

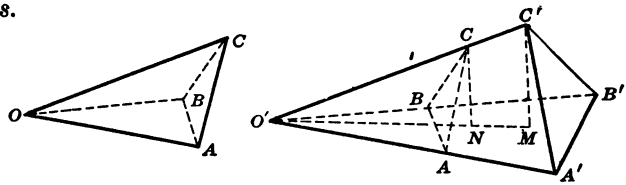


4. If the edge of a regular octaedron is e, show that its volume is given by the formula $V=\frac{1}{3}e^3\sqrt{2}$.

5. Compare the volumes of a regular tetraedron, a cube, and a regular octaedron, each 2 in. on an edge.

6. Find the area and volume of a regular tetraedron having an altitude of 8 in.

758. Theorem. *The volume of two tetraedrons that have a triedral angle of one congruent to a triedral angle of the other are to each other as the products of the three edges of these triedral angles.*



Given the tetraedrons $O\text{-}ABC$ and $O'\text{-}A'B'C'$, with triedral $\angle O$ = triedral $\angle O'$.

To prove $\frac{V}{V'} = \frac{OA \cdot OB \cdot OC}{O'A' \cdot O'B' \cdot O'C'}$, where V and V' denote the volumes of the tetraedrons.

Proof. Place tetraedron $O\text{-}ABC$ so that triedral $\angle O$ will coincide with triedral $\angle O'$.

Draw CN and $C'M$ perpendicular to $O'A'B'$.

CN and $C'M$ determine a plane which intersects $O'A'B'$ in $O'M$. Why?

Now $\frac{V}{V'} = \frac{O'AB \cdot CN}{O'A'B' \cdot C'M} = \frac{O'AB}{O'A'B'} \cdot \frac{CN}{C'M}$. Why?

But $\frac{O'AB}{O'A'B'} = \frac{O'A \cdot O'B}{O'A' \cdot O'B'}$. § 375

Further $\triangle O'NC$ and $\triangle O'MC'$ are similar rt. $\triangle$. Why?

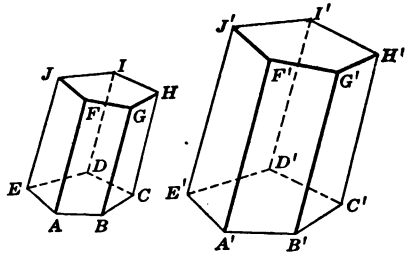
Then $\frac{CN}{C'M} = \frac{O'C}{O'C'}$. § 428

$\therefore \frac{V}{V'} = \frac{O'A \cdot O'B \cdot O'C}{O'A' \cdot O'B' \cdot O'C'} = \frac{OA \cdot OB \cdot OC}{O'A' \cdot O'B' \cdot O'C'}$. § 111

Compare this theorem and the proof with § 375.

759. Theorem. *Two parallelepipeds that have a triedral angle of one congruent to a triedral angle of the other are to each other as the products of the three edges of these triedral angles.*

760. Similar Polyedrons. Two polyedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyedral angles congruent. It is evident that the corresponding diedral angles are also equal.



761. Theorem. *The corresponding edges of similar polyedrons are proportional.*

Corresponding edges belonging to corresponding faces are proportional by § 444. Use the fact that each edge belongs to two faces and prove the theorem.

762. Theorem. *The surfaces of two similar polyedrons are to each other as the squares of any two corresponding edges.*

Given two similar polyedrons P and P', with corresponding faces A and A', B and B', C and C', $\cdots$. Also given a and a', b and b', c and c', $\cdots$, corresponding sides of these faces respectively.

To prove $\dfrac{A+B+C+\cdots}{A'+B'+C'+\cdots}=\dfrac{n^2}{n'^2}$, where n and n' are any two corresponding edges.

Proof. $\dfrac{A}{A'}=\dfrac{a^2}{a'^2}, \dfrac{B}{B'}=\dfrac{b^2}{b'^2}, \dfrac{C}{C'}=\dfrac{c^2}{c'^2}, \cdots$. § 446

But $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}=\cdots=\dfrac{n}{n'}$.

And hence $\dfrac{a^2}{a'^2}=\dfrac{b^2}{b'^2}=\dfrac{c^2}{c'^2}=\cdots=\dfrac{n^2}{n'^2}$. Why?

Therefore $\dfrac{A}{A'}=\dfrac{B}{B'}=\dfrac{C}{C'}=\cdots=\dfrac{n^2}{n'^2}$. Why?

$\therefore \dfrac{A+B+C+\cdots}{A'+B'+C'+\cdots}=\dfrac{n^2}{n'^2}$. § 403

763. Theorem. *The volumes of two similar tetraedrons are to each other as the cubes of any two corresponding edges.*

Use §§ 746, 761.

764. Theorem. *The volumes of two similar polyedrons are to each other as the cubes of any two corresponding edges.*

This theorem will not be proved. It is accepted as true since it has useful applications.

EXERCISES

1. What is the ratio of the volumes of two cubes that are 5 in. and 4 in. on an edge? What is the ratio of their areas?

2. The edges of a parallelepiped are 10, 12, and 14. Find the edges of a similar parallelepiped having $\frac{1}{9}$ as great an area. Find the ratio of their volumes.

3. Two pyramids are cut from the same pyramidal surface. The lateral edges of one are 9 in., 12 in., and 14 in. and of the other 18 in., 21 in., and 20 in. Find the ratio of their volumes.

4. Find the ratio of the volume of a regular octaedron 6 in. on an edge to the volume of a regular octaedron having half as great an area.

5. The center of gravity of a tetraedron is $\frac{1}{4}$ its altitude above the base. Find the center of gravity of a regular tetraedron 8 in. on an edge.

6. The edge of a regular tetraedron is e. Find the edge of a regular tetraedron that has a volume n times as great as the volume of the given tetraedron.

7. Find the edge of a regular tetraedron such that its volume multiplied by $\sqrt{2}$ is 288.

8. Find the volume of a regular tetraedron that has an altitude of 17 in.

9. A section of a tetraedron by a plane parallel to two opposite edges is a parallelogram.

10. The midpoints of the edges of a regular tetraedron are the vertices of a regular octaedron.

11. Two similar polyedrons have volumes of 121.5 cu. in. and 4.5 cu. in., respectively. An edge of the smaller is $1\frac{1}{2}$ in. Find the corresponding edge of the larger. *Ans.* 4.5 in.

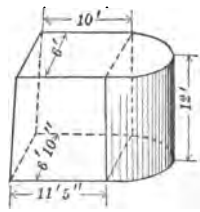
12. The area of the entire surface of a polyedron is 108 sq. in., and its volume is 432 cu. in. If the area of the entire surface of a similar polyedron is 75 sq. in., find its volume. *Ans.* 250 cu. in.

GENERAL EXERCISES

1. If from any point within a regular tetraedron perpendiculars are drawn to its faces, their sum equals an altitude of the tetraedron. (Compare this with Ex. 2, p. 96.)

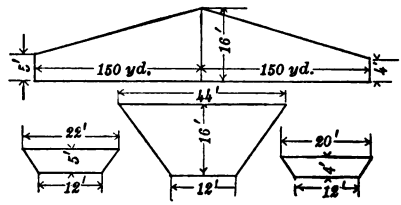
2. A wedge whose altitude is 10 in. and edge 4 in., has a base that is a square having a perimeter of 36 in. Find the volume of the wedge.

3. A concrete pier for a railway bridge has dimensions as shown in the figure, the bases being rectangles with semicircles. Find the number of cubic yards of concrete in the pier. *Ans.* 37.8 − cu. yd.



4. A railroad cut has the dimensions given in the figure, which shows the vertical section and three cross sections, one at each end and one in the middle. Find the number of cubic yards of earth removed in digging the cut.

Ans. $7972\frac{2}{3}$ cu. yd.



SUGGESTION. The part on each side of the center is a prismatoid, and can be computed separately.

5. A tank of reinforced concrete is 160 ft. long, 100 ft. wide, and 10 ft. 6 in. deep, outside dimensions. The side walls are 8 in. thick at the top and 18 in. thick at the bottom, with the slope on the inside. The bottom is 6 in. thick. Find the number of cubic yards of cement used in building the tank, and the capacity of the tank in gallons.

6. The vertices of one regular tetraedron are the centers of the faces of another regular tetraedron. Find the ratio of the volumes of the two tetraedrons.

7. The three planes passing through the lateral edges of a triangular pyramid, and bisecting the base edges, meet in a common straight line.

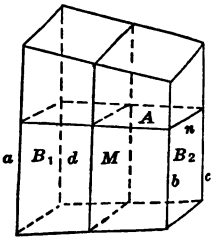
8. The six planes passing through the edges of a tetraedron, and bisecting the opposite edges, meet in a common point.

9. The point of intersection of the planes in the preceding exercise is the center of gravity of the tetraedron. Prove that the center of gravity of a tetraedron divides the line from a vertex to the center of gravity of the opposite face in parts that are in the ratio of 3 : 1.

The center of gravity of a face is at the intersection of the medians of that face.

10. The corresponding edges of three similar tetraedrons are 3 in., 4 in., and 5 in., respectively. Find the corresponding edge of a similar tetraedron equal in volume to their sum.

11. By means of the prismatoid formula show that the volume of a truncated quadrangular prism whose opposite faces are parallel, is equal to the area of a right section times one-fourth the sum of the lateral edges.



SUGGESTION. Consider two of the parallel faces, as B_1 and B_2, as bases. Then M is the midsection between these.

Let A be the area of a right section having a side in B_1 equal to n, and altitude h, the distance between B_1 and B_2.

Then $B_1 = \frac{1}{2}(a+d)\,n$, $B_2 = \frac{1}{2}(b+c)\,n$, and $M = \frac{1}{2}\left\{\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\right\}n$

$\therefore V = \frac{1}{6}h\left[\frac{1}{2}(a+d)\,n+\frac{1}{2}(b+c)\,n+2\left\{\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\right\}n\right]$

$= \frac{1}{4}hn\,(a+b+c+d) = \frac{1}{4}A\,(a+b+c+d)$.

12. The bottom of a bin of wheat is a rectangle 5 ft. by 12 ft. The top of the wheat is in a plane such that the depths at the corners are 6 ft., 5 ft., 3 ft., and 4 ft. respectively. Find the number of bushels in the bin if 1 bu. $= 1\frac{1}{4}$ cu. ft.

CHAPTER X

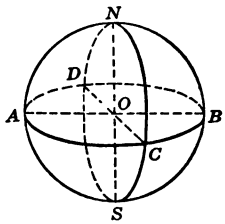
THE SPHERE AND POLYEDRAL ANGLES

GENERAL PROPERTIES

765. Sphere. A **sphere** is a closed surface all points of which are equally distant from a point within, called the **center** of the sphere.

A sphere is usually designated by a single letter at its center.

The **radius** of a sphere is the straight line connecting the center to a point in the surface. A straight line connecting two points in the surface and also passing through the center is a **diameter.**



A sphere separates space into two parts so that any point that does not lie in the surface lies within the sphere or outside it.

The distance from the center to a point within a sphere is less than the length of the radius; while a point outside a sphere is at a greater distance from the center than the length of the radius.

766. Great Circle of a Sphere. It is evident from the definitions of a sphere and of a circle that a plane passing through the center of sphere intersects the sphere in a circle whose radius is equal to the radius of the sphere. Such a circle is called a **great circle** of the sphere.

767. The following facts follow readily from the definitions given:

(1) *Radii of the same sphere, or equal spheres, are equal.*

(2) *Diameters of the same sphere, or equal spheres, are equal.*

(3) *Spheres having equal radii, or equal diameters, are equal.*

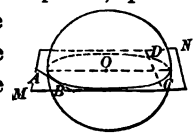
(4) *All great circles of a sphere are equal.*

(5) *The points of intersection of any two great circles of a sphere are in a diameter.*

(6) *A great circle of a sphere bisects the sphere.*

(7) *A sphere may be generated by the revolution of a semicircle about its diameter.*

768. The Straight Line and the Sphere. In order to study the relative positions of a straight line and a sphere, pass a plane through the line and the center of the sphere. The line will then have the same relations to the sphere that it bears to the great circle in this plane.



(1) *If the line intersects the great circle it cuts the circle in two points, and likewise the sphere.*

(2) *If the line is tangent to the great circle it has only one point in common with the sphere, and is perpendicular to the radius at this point.*

769. Definition. If a line has but one point in common with the sphere it is said to be **tangent to the sphere.**

EXERCISES

1. A line perpendicular to a radius at its extremity is tangent to the sphere.

2. The locus of all lines tangent to a sphere at a point is a plane perpendicular to a radius at its outer extremity.

3. Find the locus of the centers of all spheres tangent to a given line at a given point.

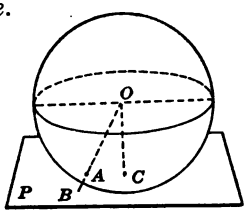
4. Find the locus of the centers of all spheres of a given radius tangent to a fixed line.

5. The plane perpendicular to a tangent at its point of contact passes through the center of the sphere.

6. What is the locus of the vertices of the right angles of the right triangles having a given hypotenuse?

770. The Plane and the Sphere. If a plane has but one point in common with a sphere, it is said to be **tangent to the sphere.**

771. Theorem. *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*



Given plane P perpendicular to the radius OC at its outer extremity C.

To prove plane P is tangent to the sphere O.

Give a proof similar to that of § 284.

772. Theorem. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

For if the plane was not perpendicular to the radius it would have a point less than the length of the radius from the center of the sphere.

773. Theorem. *If a plane is tangent to a sphere, the radius drawn to the point of tangency is perpendicular to the plane.*

EXERCISES

1. Two lines perpendicular to a radius at its outer extremity determine a plane tangent to a sphere.

2. Planes tangent to a sphere at the extremities of a diameter are parallel.

3. Explain how to construct a line tangent to a sphere at a point on the sphere. Explain how to construct a tangent plane at this point.

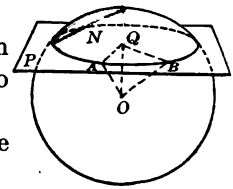
4. Explain how to construct a line and a plane tangent to a sphere and through an external point.

5. A point B is 22 in. from the center O of a sphere having a radius of 12 in. Find the distance from B to C, the point of tangency of a plane through B.

6. Find the locus of the centers of all spheres tangent to two given intersecting planes.

CIRCLES OF SPHERES

774. Theorem. ***The section of a sphere made by a plane is a circle, whose center is the foot of the perpendicular from the center of the sphere to the plane.***



Given the sphere O cut by the plane P in the section ABN, also OQ perpendicular to plane P.

To prove that section ABN is a circle with center Q.

Proof. Draw QA and QB to any two points A and B on the section ABN. Also draw OA and OB.

$\triangle AQO$ and $\triangle BQO$ are congruent right triangles. Why?

Then $AQ = BQ$. That is, any two points on section ABN are equidistant from Q. Why?

$\therefore$ section ABN is a circle with center Q. Why?

EXERCISES

1. If a plane that intersects a sphere is gradually moved from the center, describe the change in the intersection of the plane with the sphere.

2. A line that is tangent to a sphere is tangent to each section of the sphere formed by a plane containing the line.

3. The radius of a sphere is 14 in. Find the area of a section made by a plane 10 in. from the center.

4. The radius of a sphere is 12 in. If the area of a section made by a plane is 314.16 sq. in., find the distance from the center of the sphere to the plane.

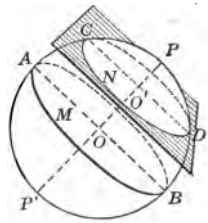
5. How far from the center of a sphere, having a radius of 12 in., must a plane pass so that the section made by this plane shall be one-half the area of a great circle of the sphere?

6. In the same sphere or equal spheres, two sections equally distant from the center, are equal, and conversely. Compare with § 281.

7. What is the locus of the projections of a given point upon the planes passing through another point?

8. What is the locus of a point from which three planes can be drawn tangent to a given sphere and form a triedral angle all of whose diedral angles are right?

775. Definitions. A **small circle of a sphere** is the intersection of the sphere with any plane not passing through its center.



The **axis of a circle** of a sphere is the diameter of the sphere that is perpendicular to the plane of the circle.

The **poles of a circle** of a sphere are the extremities of the axis of the circle.

In the figure, AMB is a great circle, CND is a small circle, PP' is the axis of each of these circles, and P and P' are the poles.

776. Theorem. *Through any three points on a sphere one and only one circle of the sphere can be drawn.*

For the three points determine a plane.

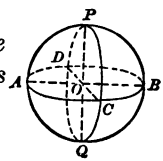
777. Theorem. *Through any two points on a sphere a great circle can be drawn.*

What third point is in the plane of the great circle?

How are the points situated when only one great circle can be drawn? How when more than one can be drawn?

778. Theorem. *Parallel circles of a sphere have the same axis and the same poles.*

779. Theorem. *If one great circle of a sphere is perpendicular to another, either circle contains the axis and poles of the other; and conversely.*



Use §§ 767 (5), 594.

EXERCISES

1. If the surface of the earth is considered a sphere, what kind of circles are the equator, the meridians, and the parallels of latitude? What is the axis of the equator? Of the parallels of latitude? What are the poles of each of these?

2. Find the circumference of the parallel of latitude 45° north of the equator. Of the parallel of latitude 60° north. (Use 4000 miles as the radius of the earth.)

DISTANCE ON A SPHERE

780. By the **distance between two points** on a sphere is meant the length of the minor arc (§ 264) of the great circle through the two points. That this is the shortest path on the sphere between the two points can be proved.

If the two points are at the extremities of a diameter, the distance between them is a semicircumference of the great circle.

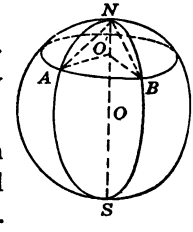
It should be noted that an arc of a great circle of a sphere here takes the place of a straight line in a plane in determining the distance. That is, the distance between two points in a plane is measured along the straight line joining them, while on a sphere the distance is measured along a great circle.

781. A **quadrant** of a sphere is one quarter of the circumference of a great circle of the sphere.

782. Polar Distance. The **polar distance** of a small circle of a sphere is the distance on the sphere from the nearer pole to a point in the circle.

The polar distance of a great circle may be taken from either of its poles to a point in the circle.

783. Theorem. ***The polar distances of a circle of a sphere are equal.***



Given $\odot O'$ on sphere O with poles N and S.

To prove that all points in $\odot O'$ are equally distant from N, or S.

Proof. Take A and B any two points in $\odot O'$, and draw great circles through NAS and NBS; also draw lines AO', BO', AN, and BN.

Prove $\triangle AO'N \cong \triangle BO'N$, and therefore $AN = BN$.

Then $\overset{\frown}{AN} = \overset{\frown}{BN}$. Why?

In all points in $\odot O'$ are equally distant from N.

$\therefore$ like manner prove for pole S.

784. Theorem. *On the same sphere or on equal spheres, the polar distances of equal circles are equal.*

785. Theorem. *The polar distance of a great circle is a quadrant; and conversely.*

786. Theorem. *The straight lines joining the points in the circle of a sphere to a pole are equal.*

EXERCISES

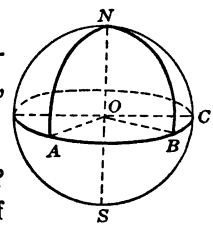
1. What is the name of the circle on the surface of the earth at a quadrant's distance from the north pole? At $23\frac{1}{2}°$ from the north pole? At $23\frac{1}{2}°$ from the south pole? At $23\frac{1}{2}°$ from the equator?

2. Show how to construct a circle on a sphere and at a given distance from a pole. Discuss for a small circle and for a great circle.

3. Show how to construct a circle of given radius on the surface of a sphere.

4. If the radius of a sphere is 10 in., find the radius of a circle having a polar distance of 45°. Of 30°.

787. Theorem. *If a point on a sphere is at a quadrant's distance from each of two other points, not the extremities of a diameter, on the sphere, it is a pole of the great circle through these two points.*



Given a point N on a sphere O, at a quadrant's distance from each of points A and B, and ABC the great circle through A and B.

To prove that N is the pole of $\odot ABC$.

Proof. $\angle AON$ and $\angle BON$ are rt. $\angle$s. Why?

Then ON is perpendicular to plane of $\odot ABC$. § 571

And SN is the axis of $\odot ABC$. Why?

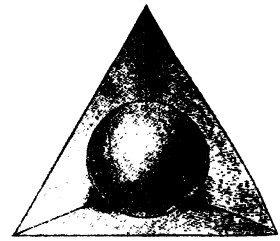
$\therefore N$ is a pole of $\odot ABC$. Why?

EXERCISES

1. Show how to find the pole of a great circle on a sphere.

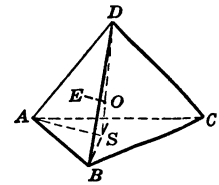
2. Show how to construct a great circle through two points on a sphere.

788. Circumscribed Polyedrons. A polyedron is said to be **circumscribed about a sphere** when the sphere is tangent to each face of the polyedron. The sphere is then said to be **inscribed in the polyedron.**



789. Inscribed Polyedrons. A polyedron is said to be **inscribed in a sphere** when all its vertices lie in the sphere. The sphere is then said to be **circumscribed about the polyedron.**

790. Theorem. *One, and only one, sphere can be inscribed in a given tetraedron.*



Given the tetraedron $ABCD$.

To prove that one sphere and only one can be inscribed in $ABCD$.

Outline of proof. Construct the planes bisecting the diedral angles AD and BD. These planes will intersect in a line DS. Why?

Then DS is the locus of all points equally distant from faces ADC, ABD, and BDC. Why?

Construct the plane bisecting the diedral angle AB.

This plane will intersect DS in some point O. Why?

Then O is equally distant from the four faces of the tetraedron. Why?

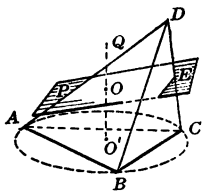
If a sphere is constructed with O as center and the perpendicular from O to any face, as OE, for radius, this sphere will be tangent to each face of the tetraedron, and therefore is an inscribed sphere. § 789

To show that only one inscribed sphere can be constructed show that there is only one point O.

Compare this theorem and proof with the corresponding problem in plane geometry. (§ 325).

EXERCISE. Find the locus of the centers of all spheres tangent to the faces of a triedral angle.

791. Theorem. *One, and only one, sphere can be circumscribed about a given tetraedron.*



Given the tetraedron $ABCD$.

To prove that one sphere and only one can be circumscribed about $ABCD$.

Outline of proof. Circumscribe a circle about $\triangle ABC$, and erect a perpendicular $O'Q$ to its plane at its center O'.

Then $O'Q$ is the locus of all points equally distant from A, B, and C. Why?

At the midpoint E of the edge CD construct a plane P perpendicular to CD.

Then plane P is the locus of all points equally distant from C and D. Why?

But plane P intersects $O'Q$ in a point O. Why?

Then O is equally distant from the four vertices of the tetraedron, and a sphere having O as center and OA as radius will pass through the four vertices A, B, C, and D. Why?

Therefore a sphere can be circumscribed about the tetraedron.

To show that only one sphere can be circumscribed about $ABCD$, show that there is only point O.

Compare this theorem and proof with the corresponding problem in plane geometry. (§§ 316, 317).

EXERCISES

1. What is the locus of points equidistant from four points not all in the same plane?

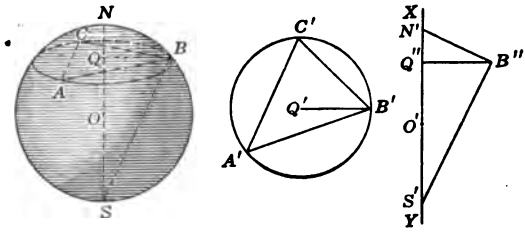
2. How many points are necessary to determine a sphere? How many points on a sphere are necessary to determine it if its center is given?

3. Prove that the planes that are the perpendicular bisectors of the edges of a tetraedron pass through a common point.

4. Prove that, if a circle is circumscribed about each face of a tetraedron, the perpendiculars to the respective faces at the centers of the circumscribing circles are concurrent.

5. One, and only one, sphere can be inscribed in or circumscribed about a cube.

792. Problem. *To find the radius of a given material sphere.*



Given a material sphere O.

To find its radius.

Construction. Choose any point N on the sphere as a pole, and describe a convenient circle ABC.

Choose any three points on this circle as A, B, and C.

Construct in a plane $\triangle A'B'C'$ congruent to $\triangle ABC$. § 25

Circumscribe a circle with center Q' about $\triangle A'B'C'$.

Draw $Q''B''$ equal to the radius $Q'B'$, and then construct $XY \perp Q''B''$ and through Q''.

With the compasses lay off $B''N'$ equal to BN, to meet XY at N', and construct $B''S' \perp B''N'$ at B'' and extend it to meet XY at S'.

Determine the bisecting point O' of $S'N'$.

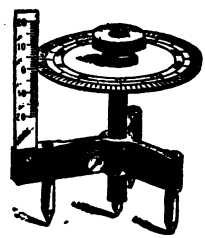
Then $O'N'$ is the radius of the sphere O.

The proof is left to the student.

EXERCISES

1. If $NB = 13$ in. and $QB = 12$ in., compute ON. *Ans.* 16.9 in.

2. By means of an instrument called a spherometer, the distances QN and QB can be measured. Show how the radius ON can then be computed. Such an instrument is used practically in determining the radius of curvature of a spherical lens.



3. Find the radius of curvature of a spherical lens, that is, the radius of the sphere of which the surface of the lens is a part, if $QN = 0.125$ cm. and $QB = 2$ cm. If $QN = 0.643$ cm. and $QB = 2.5$ cm.

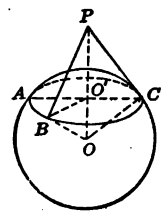
793. Sphere Inscribed in Cylinder or Cone. A sphere is said to be **inscribed in a cylinder,** or the cylinder **circumscribed about the sphere,** when the bases of the cylinder and all its elements are tangent to the sphere.

A sphere is said to be **inscribed in a cone,** or the cone **circumscribed about the sphere,** when the base of the cone and all its elements are tangent to the sphere.

794. Cylinder or Cone Inscribed in a Sphere. A cylinder is said to be **inscribed in a sphere,** or the sphere **circumscribed about the cylinder,** when its bases are circles of the sphere.

A cone is said to be **inscribed in a sphere,** or the sphere **circumscribed about the cone,** when its base is a circle of the sphere and its vertex lies on the sphere.

795. Theorem. *All the tangents to a sphere from a given point are equal, and their points of contact are in a circle of the sphere.*



Given sphere O, and point P outside the sphere.

To prove that the tangents to the sphere from the point P are equal, and that their points of contact lie in a circle.

Outline of proof. Let PB be one tangent. Then PB is tangent to a great circle determined by P, B, and O. Why?

Therefore $\angle PBO$ is a rt.$\angle$. § 288

Pass a plane through $B \perp PO$. This will intersect the sphere in a circle with center O'.

Further, a line drawn from P to any point, as C, in the $\odot O'$ is tangent to the sphere O, for $\triangle OCP \cong \triangle OBP$, and therefore $PC \perp OC$. Why?

Show that no tangent from P could have its point of contact outside of $\odot O'$.

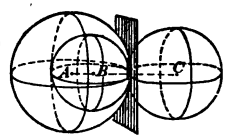
That the tangents are all equal follows from § 584.

RELATIVE POSITIONS OF SPHERES

796. The relative positions of two spheres are analogous to the relative positions of two circles. (See §§ 291-296.)

797. Line of Centers. The line joining the centers of two spheres is the line of centers.

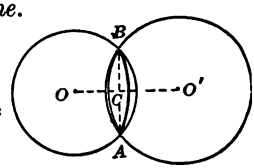
798. Tangent Spheres. If two spheres are tangent to the same plane at the same point, they are **tangent spheres.** They may be tangent internally or externally.



Spheres A and B are tangent internally; and A and C externally.

799. Concentric Spheres. Spheres that have a common center are **concentric spheres.**

800. Theorem. *The intersection of two spheres is a circle, whose center is in a straight line joining the centers of the spheres and whose plane is perpendicular to that line.*



Given two intersecting spheres O and O'.

To prove that the intersection of the spheres is a circle whose center is in OO' and whose plane is perpendicular to OO'.

Outline of proof. • Through O, O', and any point A of the intersection, pass a plane. This plane intersects the two spheres in two great circles intersecting in two points. Why?

If A and B are these points, then OO' is the perpendicular bisector of AB. § 295

Show that if the entire figure is revolved about OO', the great circles generate the spheres, and point A generates a circle with center C and radius CA, which is common to the spheres.

Also show that the plane of the circle generated by A is perpendicular to OO'.

Further, show that if any point of the intersection was outside of this circle, then the spheres would coincide. (§ 791).

801. Theorem. *If two spheres are tangent to each other, the line of centers passes through the point of contact.*

Proof similar to that of § 296.

EXERCISES

1. State and prove exercises concerning the sphere analogous to Exercises 1 to 5, page 115.

2. What is the locus of points at a distance r from a given point O, and at a distance r' from a given point O'?

3. Find the center of a sphere which contains a given circle and also contains a given point not lying in the plane of the circle.

4. Two spheres with radii 34 and 50, respectively, intersect. If the distance between their centers is 56, find the radius of their circle of intersection, and the distance from the center of each sphere to the center of the circle of intersection.

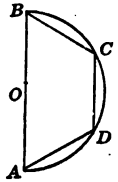
5. The line of contact of a sphere inscribed in a circular cone with the conical surface of the cone, is a small circle of the sphere.

6. The line of contact of a sphere inscribed in a circular cylinder with the cylindrical surface, is a great circle of the sphere.

MEASUREMENT OF THE SPHERE, AREA

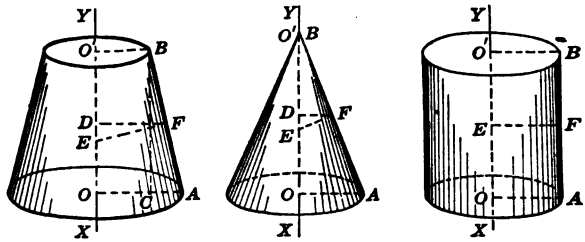
802. Area of a Sphere. By the **area of a sphere** is meant the numerical measure of the surface, that is, the number of units of area in it.

As with the cylinder and cone, a sphere is a curved surface, and a plane unit of area cannot be applied directly to it to find its numerical measure. Its area can be determined as a limit. It is based upon (7) of § 767, which states that a sphere is generated by revolving a semicircle about its diameter, and the following statement which is accepted without proof:



If in a semicircle generating a sphere, one-half of a regular polygon be inscribed so that a vertex lies at each end of the diameter, then, as the number of the sides of the inscribed semipolygon is indefinitely doubled, the area generated by the semipolygon approaches the area of the sphere as a limit.

803. Theorem. *The area of the surface generated by a segment of a straight line revolving about an axis, in its plane but not perpendicular to it and not intersecting it, is equal to the projection of the segment upon the axis multiplied by the circumference of the circle whose radius is the perpendicular erected at the midpoint of the segment and terminated by the axis.*



Given the segment AB revolving about the axis XY in its plane.

To prove that $S = OO' \times 2\pi \times EF$, where S denotes the area of the surface generated, OO' is the projection of AB upon XY, and FE is the perpendicular at the midpoint of AB.

Proof. Three cases arise:

CASE I, in which AB is oblique to and does not meet XY. The surface generated is the lateral surface of a frustum of a right circular cone.

The proof of this is given in § 729.

CASE II, in which AB is oblique to and meets XY. The surface generated is the lateral surface of a right circular cone.

$S = \pi \times OA \times AB$. § 719

Show that $\triangle DEF \backsim \triangle OAB$.

Then $AB : OB = EF : DF$, and $AB \times DF = OB \times EF$.

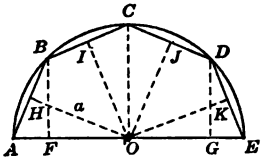
Since $OB = OO'$ and $OA = 2DF$, $AB \times OA = 2OO' \times EF$.

$\therefore S = OO' \times 2\pi \times EF$. § 111

CASE III, in which AB is parallel to XY. The area generated is the lateral surface of a right circular cylinder.

Give proof.

804. Theorem. ***The area of a sphere is equal to four times the area of a great circle of the sphere.*** $S=4\pi r^2$.



Given the area S of a sphere generated by the revolution of the semicircle O of radius r, about the diameter AE.

To prove that $S=4\pi r^2$.

Proof. Inscribe in the semicircle one-half of a regular polygon of an even number of sides, as $ABCDE$. Let A denote the apothem of the polygon, and let F, O, and G be the projections of B, C, and D respectively upon the diameter AE.

For each side, the apothem is the perpendicular bisector. Why?

Then area generated by $AB=AF\times 2\pi a$. § 803

Similarly area generated by $BC=FO\times 2\pi a$.

Etc.

Adding these equations and denoting by S' the area generated by the revolution of the semipolygon $ABCDE$,

$$S'=(AF+FO+\cdots)2\pi a.$$

But $$(AF+FO+\cdots)=AE=2r.$$

Then $S'=4\pi ra$, which is always true as the number of the sides of the polygon is continuously doubled.

Further $S'\to S$ and since $a\to r$, $4\pi ra\to 4\pi r^2$. §§ 802, 490, 485 (2)

$$\therefore S=4\pi r^2. \qquad \text{§ 485 (1)}$$

Show that other forms of the formula for the area of a sphere are $S=2\pi rd$, and $S=\pi d^2$.

805. Theorem. *The areas of two spheres are to each other as the squares of their radii; or, as the squares of their diameters.*

For, if S, and S' are the areas of two spheres,

$$\frac{S}{S'}=\frac{4\pi r^2}{4\pi r'^2}=\frac{r^2}{r'^2}, \text{ and } \frac{S}{S'}=\frac{\pi d^2}{\pi d'^2}=\frac{d^2}{d'^2}.$$

EXERCISES

1. Given the area of a sphere, to find its radius and its diameter.

2. Find the area of a sphere whose radius is 5. Of a sphere whose diameter is 16. Of a sphere whose circumference is 24 in.

3. Find the radius of a sphere whose area is 100 sq. in.

4. How many square feet of tin will it take to roof a hemispherical dome 40 ft. in diameter?

5. Find the diameter of a sphere that has the same area as a cube that is 8 in. on an edge.

6. Find the area of a sphere that is circumscribed about a cube that is $12\sqrt{3}$ in. on an edge. *Ans.* 4071.5 sq. in.

7. The number of square inches in the area of a sphere is equal to the number of linear inches in the circumference of a great circle of the sphere. Find the radius of the sphere.

8. Show that, if a cylinder is circumscribed about a sphere, the lateral area of the cylinder is equal to the area of the sphere. Also show that the area of the sphere equals two-thirds the total area of cylinder.

9. The radius of the earth is 3960 miles and the radius of the moon is 1080 miles. Find the ratio of their areas.

10. Show how to determine the distance apart the points of a compass must be placed to draw a great circle upon a material sphere that has a diameter of 14 in. So as to draw a circle having a radius of 5 in.

11. What is the locus of the centers of all spheres tangent to a given plane at a given point?

12. What is the locus of the centers of all spheres passing through three given points?

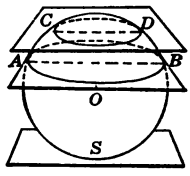
13. What is the locus of points at a distance r_1 from a point O_1 and at a distance r_2 from another point O_2? Discuss.

14. Find the area of a sphere circumscribed about a tetraedron whose edge is 4 in.

15. The radii of two intersecting spheres are 10 in. and 12 in., respectively, and the distance between their centers is 16 in. Find the radius of the circle of intersection of the two spheres.

16. Three equal spheres each tangent to the other two rest on a plane. A fourth sphere rests on the first three spheres. Find the distance from the center of the fourth sphere to the plane, if each sphere has a radius of 8 in. *Ans.* 21.064 in.

806. Zones. The portion of the surface of a sphere included between two parallel planes is called a **zone.**



It is evident that a zone can be considered as generated by an arc of a great circle.

The circles made by the two parallel planes are called the **bases** of the zone, and the distance between the planes is the **altitude** of the zone.

If one of the planes is tangent to the sphere the zone is called a **zone of one base.**

The zones on the surface of the earth are included between certain parallels of latitude. The torrid and temperate zones are zones of two bases, and the frigid zones are zones of one base.

807. Theorem. *The area of a zone of a sphere is equal to the product of the altitude of the zone and the circumference of a great circle of the sphere; that is, area* $= 2\pi rh$, *where* r *denotes the radius of a great circle and* h *the altitude of the zone.*

The proof is similar to that of § 804.

808. Theorem. *The areas of zones of the same sphere or of equal spheres are to each other as their altitudes.*

EXERCISES

1. Find the area of a zone of one base and having an altitude of 5 ft. on a sphere 8 ft. in diameter.

2. Find the radius of a sphere, on which a zone of altitude 5 in. has an area of 100 sq. in.

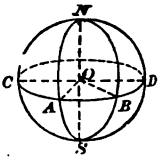
3. The diameter of a sphere is 12 in., and is divided into four equal parts by parallel planes. Find the area of each zone.

4. Find what part of the earth's surface is north of the parallel 45° north. North of the parallel 60° north.

5. Show that one-half of the earth's surface lies within 30° of the equator.

6. What part of the earth's surface could be seen from a point whose distance from the surface is equal to the radius? Twice the radius?

809. Lunes. That portion of a sphere between the halves of two great circles of the sphere is called a **lune.**



The diedral angle between the planes of the great circles bounding the lune is the **angle of the lune.**

From § 589 it follows that the angle of a lune is measured by the plane angle of the diedral angle formed by the planes of the great circles bounding the lune.

810. Comparison of Lunes. If, on the same sphere, one lune is placed on another with one bounding semicircle in common, the other bounding semicircles either coincide, or one lies wholly within the lune bounded by the other. It is evident then that:

(1) *On the same sphere or equal spheres, two lunes having equal angles are equal; and conversely.*

(2) *On the same sphere or equal spheres, two lunes are in the same ratio as their angles.*

811. Theorem. *The area of a lune is given by the formula* $L=\frac{u}{360}(4\pi r^2)$, *where L denotes the area of the lune, u the angle of the lune in degrees, and r the radius of the sphere.*

To prove this, consider the sphere as a lune whose angle is 360°, then by § 810 (2), $\frac{L}{4\pi r^2}=\frac{u}{360}$, or $L=\frac{u}{360}(4\pi r^2)$.

EXERCISES

1. Find the area of a lune whose angle is 30° on a sphere 8 ft. in diameter.

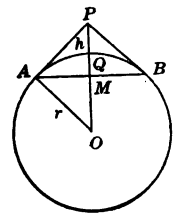
2. Find the part of the earth's surface between the equator and the parallel 30° north, and between the meridians 120° west and 150° west.

3. The chord of the polar distance of a circle is 8 in. If the radius of the sphere is 10 in., what is the radius of the circle? What is the area of the zone cut off by the circle?

4. Find the altitude of a zone whose area equals the area of a great circle of the sphere.

5. A plane passes through a sphere bisecting at right angles a radius of the sphere. Compare the areas of the two parts of the sphere.

6. If h is the height of P above the surface of the earth and r the radius, then S, the extent of the surface of the earth visible from P, is given by the formula $S=\dfrac{2\pi r^2 h}{r+h}$.



§ 807

$S=2\pi r\times MQ$.

But $MQ=r-OM$, and $OM : OA=OA : OP$.

Then $OM=\dfrac{r^2}{r+h}$, and $MQ=r-\dfrac{r^2}{r+h}=\dfrac{rh}{r+h}$.

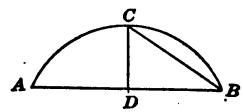
$\therefore S=2\pi r\times\dfrac{rh}{r+h}=\dfrac{2\pi r^2 h}{r+h}$.

7. How many miles above the surface of the earth is a point from which one-fourth of the surface can be seen? *Ans.* r.

8. From a point 6 ft. from a sphere one-fourth of its area is visible. Find the radius of the sphere.

9. An ash tray is in the form of the zone of a sphere. If the tray is 1 in. deep and $3\frac{1}{2}$ in. in diameter at the top, find the diameter of the blank from which it is pressed.

10. A zone of one base has the same area as a circle whose radius is the chord of the generating arc of the zone, that is, the area of the zone generated by arc BC revolving about DC is equal in area to a circle having chord BC for a radius.

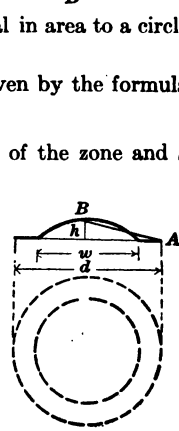


11. Show that the area of a zone of one base is given by the formula

$$S=\tfrac{1}{4}\pi(w^2+4h^2)=\pi[(\tfrac{1}{2}w)^2+h^2],$$

where S denotes the area, w the diameter of the base of the zone and h the altitude.

12. In a practical handbook, the following rule is given as *nearly* correct. Show that it is accurate. The area of a flanged spherical segment is equal to a circle of radius equal in length to the line drawn from the top of the segment to the edge of the flange, that is equal to a circle of radius AB.



SUGGESTION. Area of zone $=\pi[(\frac{1}{2}w)^2+h^2]$.

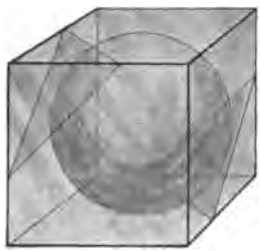
Area of flange $=\pi[(\frac{1}{2}d)^2-(\frac{1}{2}w)^2]$.

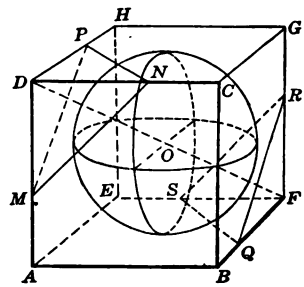
Area of zone and flange $=\pi[(\frac{1}{2}d)^2+h^2]=\pi\overline{AB}^2$.

VOLUME OF A SPHERE

812. By the **volume of a sphere** is meant the volume within the sphere.

813. If a regular polyedron, for instance a cube, is circumscribed about a sphere, and lines drawn from each vertex to the center of the sphere, these lines may be taken as the lateral edges of pyramids having their vertices at the center of the sphere and having the faces of the polyedron as bases. The volume of the polyedron can then be expressed as the volume of these pyramids.

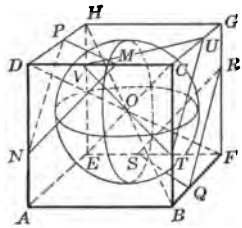




Further, if tangent planes be drawn where each lateral edge intersects the sphere, a circumscribed polyedron of a greater number of faces will be formed. It is evident that this process can be continued indefinitely, thus forming polyedrons whose areas and volumes steadily decrease, and can be made to differ as little as desired from the area and volume, respectively, of the sphere. The following statement can then be accepted.

814. *If each vertex of a circumscribed polyedron is joined to the center of the sphere, and tangent planes are drawn at the points where these lines intersect the sphere, then, as the number of faces is increased indefinitely, the area and the volume of the polyedron approach as limits the area and the volume, respectively, of the sphere.*

815. Theorem. ***The volume of a sphere is equal to the product of its area by one-third of its radius.***



Given a sphere O of radius r and area S.

To prove that the volume $V = S \times \frac{1}{3}r$.

Proof. Circumscribe about the sphere any polyedron, for instance a cube, and denote the surface of the polyedron by S'.

Since the polyedron may be considered as composed of pyramids having the faces of the polyedron as bases and the center of the sphere as common vertex, the volume V' of the polyedron is $V' = S' \times \frac{1}{3}r$.

If the number of faces of the polyedron is indefinitely increased as described in § 813,

$V' \to V$, $S' \to S$, and $S' \times \frac{1}{3}r \to S \times \frac{1}{3}r$. §§ 814, 485 (2)

But $V' = S' \times \frac{1}{3}r$ is always true as the number of faces is increased.

$\therefore V = S \times \frac{1}{3}r$. § 485

816. Theorem. *The volume V of a sphere of radius r and diameter d is given by the formulas: $V = \frac{4}{3}\pi r^3$ and $V = \frac{1}{6}\pi d^3$.*

For $V = \frac{1}{3}rS$ by § 815. But $S = 4\pi r^2$ or πd^2 by § 804.

$\therefore V = \frac{4}{3}\pi r^3$, and $V = \frac{1}{6}\pi d^3$.

817. Theorem. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

Let V_1, r_1, d_1 and V_2, r_2, d_2 be the volumes, radii and, diameters, respectively.

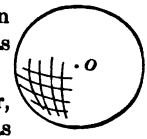
Then $\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{r_1^3}{r_2^3}$, or $\frac{V_1}{V_2} = \frac{\frac{1}{6}\pi d_1^3}{\frac{1}{6}\pi d_2^3} = \frac{d_1^3}{d_2^3}$.

818. Theorem. *The prismoid formula holds for a sphere.*

In the formula, $V=\frac{1}{6}h(B_1+B_2+4M)$ of § 751, $h=2r$, $B_1=0$ $B_2=0$, $4M=4\pi r^2$. $\therefore V=\frac{4}{3}\pi r^3$.

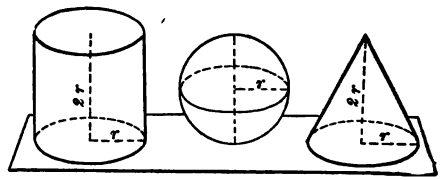
819. Theorem. *The volume of a sphere is equal to two-thirds the volume of the circumscribed right cylinder.*

EXERCISES



1. Consider the surface of a sphere divided as shown in the figure. Can you conclude that the volume is as given in § 815?

2. Show that the volumes of the right circular cylinder, the sphere, and the right circular cone with dimensions as given in the figure, are in the ratio of 3 : 2 : 1.



NOTE. Archimedes (287-212 B.C.) who was the greatest mathematician of antiquity, proved the theorem of Exercise 2. This theorem may well be regarded as one of the most beautiful in elementary mathematics. Archimedes proved many of the most important theorems in regard to the sphere.

3. State a formula for finding the radius of a sphere when the volume is given. One for finding the diameter when the volume is given.

4. A ball of diameter 2 in. is inside a right circular cylinder of diameter and altitude each 2 in. How many cubic inches of water can be poured into the cylinder?

5. Find the weight of a spherical cast iron ball of diameter 6 in., if cast iron weighs 0.26 lb. per cubic inch.

6. Find the volume of a hollow spherical shell, the outer diameter being 8 in. and the thickness 2 in. *Ans.* 234.6 cu. in.

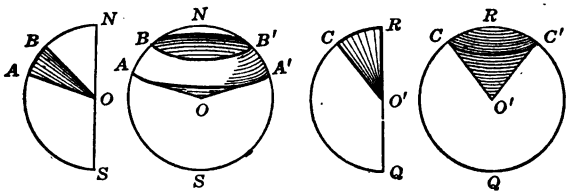
7. How many spherical bullets $\frac{2}{5}$ of an inch in diameter can be made from 5 pounds of lead, if lead weighs 0.412 pounds per cubic inch?

8. The volume of two spheres are in the ratio of 512 : 27. Find the radius of each if the sum of their radii is 44 in.

9. A water tank, having a total length of 6 ft. and a diameter of 18 in., is in the form of a right circular cylinder with two hemispherical ends. Find its capacity in gallons. *Ans.* 72.7+ gal.

820. Spherical Sector. If a sphere is generated by the revolution of a semicircle, the solid generated by a sector whose arc is a part of this semicircle is called a **spherical sector.**

A spherical sector generated by a circular sector revolving about one of its bounding radii is called a **spherical cone.**



The zone generated by the arc of the generating sector is called the **base** of the spherical sector.

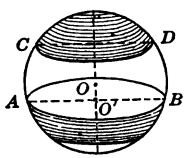
821. Theorem. *The volume of a spherical sector is equal to one-third the product of the area of its base, and its radius.*

EXERCISES

1. A right circular cone having a vertex angle of 60° has its vertex at the center of a sphere 4 ft. in diameter. What is the volume cut from the sphere by the cone?

2. Find the volume of a spherical sector cut from a sphere 4 ft. in diameter, and whose base is a zone having an altitude of 8 in.

822. Spherical Segment. The portion of the volume of a sphere contained between two parallel planes that intersect the sphere or are tangent to it, is called a **spherical segment.**

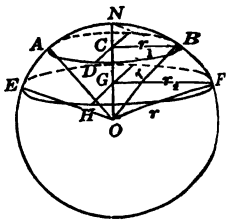


The sections of the sphere made by the two parallel planes are the **bases** of the spherical segment.

If one of the parallel planes is tangent to the sphere, the segment is called a **spherical segment of one base.**

The distance between the parallel planes is the altitude of the spherical segment.

823. Theorem. *The volume of a spherical segment is given by the formula* $V=\frac{1}{2}\pi h(r_1^2+r_2^2)+\frac{1}{6}\pi h^3$, *where* h *denotes the altitude, and* r_1 *and* r_2 *the radii of the bases of the segment.*



Given the spherical segment with altitude h, and bases ADB and EHF having radii r_1 and r_2 respectively.

To prove $V=\frac{1}{2}\pi h(r_1^2+r_2^2)+\frac{1}{6}\pi h^3$.

Proof. Let $OG=x$,

$V=$ volume of spherical sector formed by revolving $OBF+$ volume of cone $O\text{-}ADB$ $-$ volume of cone $O\text{-}EHF$. Why?

(1) $V=\frac{2}{3}\pi r^2h+\frac{1}{3}\pi r_1^2(h+x)-\frac{1}{3}\pi r_2^2x$. Why?

To reduce to the form desired it is necessary to eliminate r and x, and this requires two other equations. They are:

(2) $r^2=r_1^2+(h+x)^2$, and (3) $r^2=r_2^2+x^2$. Why?

From (2) and (3) $x=\dfrac{r_2^2-r_1^2-h^2}{2h}$.

Substituting this value of x in (3),

$$r^2=\frac{h^4+r_1^4+r_2^4+2r_1^2h^2+2r_2^2h^2-2r_1^2r_2^2}{4h^2}.$$

Substituting these values of x and r^2 in (1) and simplifying,

$$V=\tfrac{1}{2}\pi h(r_1^2+r_2^2)+\tfrac{1}{6}\pi h^3.$$

Remark. The same formula is derived if the center of the sphere is within the spherical segment.

If the spherical segment has but one base, $r_1=0$, and the formula becomes $V=\frac{1}{2}\pi hr_2^2+\frac{1}{6}\pi h^3$.

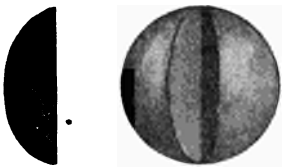
The student should carry out the work in each of these cases.

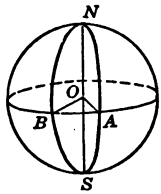
EXERCISES

1. What part of the volume of the earth is between the planes cutting its surface in the parallels of 30° and 45° north latitude? What part is north of a plane cutting the surface in the parallel of 60° north latitude?

2. In the formula for the volume of a spherical segment, solve for each letter so far as possible.

824. Spherical Wedge. That portion of the volume of a sphere contained between the planes of two great semicircles is called a **spherical wedge.** Its curved surface is its **base,** and is, evidently, a lune.





The solid *A-NS-B* is a wedge. The lune between the semicircles *NAS* and *NBS* is the base of the wedge. The angle *AOB* of the wedge is the angle of its lune.

825. Theorem. *The volume of a spherical wedge is equal to one-third the product of the area of its base and its radius.* That is, $V = \frac{u}{360}(\frac{4}{3}\pi r^3)$, *where r denotes the radius of the sphere and u the angle of the lune in degrees.*

By a discussion similar to that for the volume of a sphere, the volume of a wedge is given by the formula $V = \frac{1}{3}Lr$, where L is the area of the lune.

But $L = \frac{u}{360}(4\pi r^2)$ by § 811. $\therefore V = \frac{u}{360}(\frac{4}{3}\pi r^3)$.

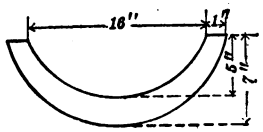
EXERCISES

1. Find the volume of a spherical wedge whose angle is 20°, cut from a sphere whose radius is 12 in.

2. Find the angle of a spherical wedge if its volume is 418.88 cu. in., and the radius of the sphere is 10 in.

3. A circular flower bed is in the form of a spherical segment that is 25 ft. in diameter and $2\frac{1}{2}$ ft. in altitude. How many loads of dirt did it take to build it up if one load is $1\frac{1}{2}$ cu. yd.?

4. The figure is a vertical cross section of a casting, the inner and outer "skins" being spherical zones. Find the weight of metal at 0.35 lb. per cubic inch necessary to make the casting. *Ans.* 176 lb.



5. Find the volume of the segment between two parallel planes 6 in. apart that cut a sphere having a radius of 12 in., if one plane passes 2 in. from the center. There are two cases: (1) when the center of the sphere lies outside the segment, and (2) when the center lies in the segment. *Ans.* (1) 2186.6 cu. in.; (2) 2638.9 cu. in.

6. The number of square inches in the area of a certain sphere is equal to four times the number of cubic inches in its volume. Find the radius of the sphere.

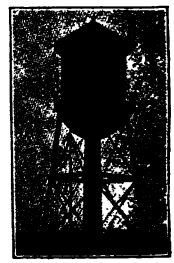
7. A copper ball to be used as a float must contain 22,449.4 cu. in. What must be its diameter? *Ans.* 35 in.

8. Derive the formula for the volume of a sphere from the formula for the volume of a spherical sector.

9. Derive the formula for the volume of a sphere from the formula for the volume of a spherical segment.

SUGGESTION. Let $r_1 = 0$ and $r_2 = r$ and obtain the volume of a hemisphere.

10. If oranges $2\frac{1}{2}$ in. in diameter are selling for 30 cents a dozen, what should be the price for oranges 3 in. in diameter?



11. The water tank shown in the figure is in the form of a right circular cylinder with a hemisphere at the bottom. Find the capacity in gallons if the diameter is 22 ft. and the altitude of the cylinder is 24 ft. 6 in.

12. Find how many square feet of sheet steel in the tank if it is open at the top.

13. The roof over the tank is third pitch and projects 2 ft. at the eaves. Find the number of square feet in it.

14. A shop received an order for 100 twelve-inch steel balls to be used in crushing cement in a cement mill. They were made from $8\frac{1}{2}$-inch round steel shafting. What length of shaft was taken for each ball if 5% was allowed for scale?

15. A round or button-head machine screw has a head in the form of a spherical segment of one base. Show that the volume of the head is given by the formula: $V = \pi h(\frac{1}{8}w^2 + \frac{1}{6}h^2)$, where w is the diameter of the head and h its height.

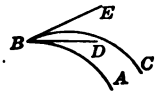
16. Find the volume of the head of a round head machine screw if the diameter of the head is 0.731 in. and the height is 0.279 in.

Ans. 0.070 cu. in.

SPHERICAL ANGLES

826. Angles Formed by Arcs. The angle formed by two arcs of circles is the plane angle formed by the lines tangent to the arcs at the point of intersection.

Thus the angle formed by the arcs AB and CB is the angle formed by the tangent lines DB and EB.

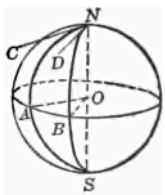


According to this definition, any two intersecting circles on a sphere form angles. In this text, however, only those angles formed by great circles are considered.

827. Spherical Angle. The angle between two great circle arcs is called a **spherical angle.** The point where the two arcs meet is the **vertex** of the angle, and the arcs are the **sides** of the angle.

The terms *right*, *acute*, and *obtuse* are applied to spherical angles and have the same meaning as with plane angles.

828. Since the planes of two great circles intersect in a diameter (§ 767), and the tangents at the points of intersection are perpendicular to this diameter (§ 288), the angle formed by the tangents is the plane angle of the diedral angle formed by the planes of the great circles § 588. It follows then from § 826 that:



A spherical angle equals in degrees the diedral angle between the planes of its sides.

Spherical $\angle ANB = \angle C\dot{N}D =$ °diedral $\angle A\text{-}NS\text{-}B$.

EXERCISES

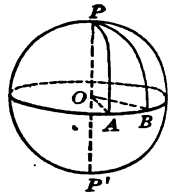
1. Form definitions for the following when applied to spherical angles: adjacent, complementary, supplementary, vertical.

2. Prove that the sum of all the spherical angles about a point on a sphere is equal to 360°.

3. At what angle does a meridian of the earth intersect the equator?

4. Prove that vertical spherical angles are equal.

829. Theorem. ***A spherical angle is equal in degrees to the arc of the great circle described from its vertex as a pole, and included between its sides, produced if necessary.***



Given spherical $\angle APB$, $\overset{\frown}{AB}$ an arc of a great circle whose pole is P, and which is included between the sides AP and BP.

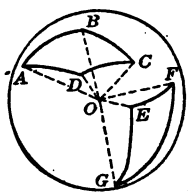
To prove that spherical $\angle APB = {}^\circ\overset{\frown}{AB}$.

Proof. Draw PO, AO, and BO.

PO is perpendicular to plane of $\overset{\frown}{AB}$.	§ 775
Then $OA \perp PO$ and $BO \perp PO$.	Why?
And $\angle AOB$ is the plane angle of diedral $\angle A\text{-}OP\text{-}B$.	Why?
Hence diedral $\angle A\text{-}OP\text{-}B = {}^\circ\angle AOB$.	§ 589 (3)
But $\angle AOB = {}^\circ\overset{\frown}{AB}$.	§ 305
$\therefore$ spherical $\angle APB = {}^\circ\overset{\frown}{AB}$.	§ 111

SPHERICAL POLYGONS AND POLYEDRAL ANGLES

830. Spherical Polygons. A portion of a sphere bounded by three or more arcs of great circles is called a **spherical polygon.** The bounding arcs are the **sides** of the polygon, the spherical angles formed by the sides are the **angles** of the polygon, and their vertices the **vertices** of the polygon.



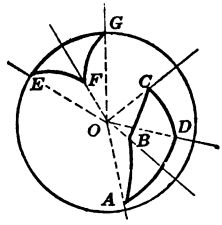
831. A **diagonal** of a spherical polygon is the arc of a great circle joining any two vertices not adjacent.

It is evident that a side of a spherical polygon can be greater than a semicircle, but in this text only those will be considered whose sides are each less than a semicircle.

832. A **spherical triangle** is a spherical polygon of three sides.

The terms *right, acute, obtuse, isosceles,* and *equilateral* are applied to spherical triangles and have the same meaning as with plane triangles. It will be found, however, that a spherical triangle may have one, two, or three right or obtuse angles.

833. Central Polyedral Angles. It follows from the definition of a spherical polygon that the planes of the sides of the polygon pass through the center of the sphere. These planes form a polyedral angle, called the **corresponding central polyedral angle** of the spherical polygon. This polyedral angle and the spherical polygon are so related that it is convenient to consider them in connection with each other.



It follows directly from §§ 305 and 829 that:

(1) *The sides of a spherical polygon are equal in degrees to the face angles of the polyedral angle.*

(2) *The angles of a spherical polygon are equal in degrees to the diedral angles of the polyedral angle.*

Because of these relations it follows that to each property of a polyedral angle there is a corresponding property of the spherical polygon, and *vice versa*. One of these properties can be stated when the other is given by making the substitution indicated in the following:

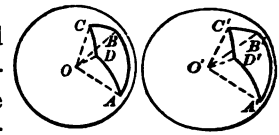
Spherical Polygon	Polyedral Angle
Sides	Face angles
Angles of polygon	Diedral angles
Vertices of polygon	Edges
Center of sphere	Vertex

In this manner a theorem relating to spherical polygons can readily be worded so as to apply to polyedral angles, and *vice versa*.

834. A **convex spherical polygon** is a spherical polygon no side of which, when produced, will enter the polygon.

It is evident that the corresponding central polyedral angle of a convex spherical polygon is a convex polyedral angle. (See § 605.)

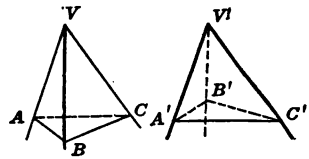
835. Two spherical polygons on the same sphere or equal spheres are **congruent** if their corresponding central polyedral angles are congruent. (See § 607.)



The spheres O and O' are equal and the polyedral angles having their vertices at the centers of the spheres are congruent. Then the corresponding spherical polygons $ABCD$ and $A'B'C'D'$ are congruent.

836. Symmetric Polyedral Angles and Spherical Polygons. Two polyedral angles are **symmetric** if the face angles and diedral angles of one are respectively equal to those of the other but arranged in reverse order.

The triedral angles V-ABC and V'-$A'B'C'$ are symmetric, for diedral angles AV, BV, and CV are equal respectively to diedral angles $A'V'$, $B'V'$, and $C'V'$; and the face angles AVB, BVC, and CVA are equal respectively to face angles $A'V'B$, $B'V'C'$, and $C'V'A'$.

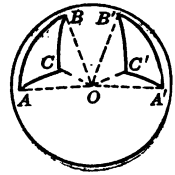


It is to be noticed that if the triedral angles are viewed from the vertices V and V', then the parts of the triedral angle V-ABC are arranged in the reverse order to the parts of V'-$A'B'C'$.

If the two triedral angles were made of pieces of cloth sewed together, then, if one were turned "inside out," they could be made to coincide. They would then be congruent polyedral angles.

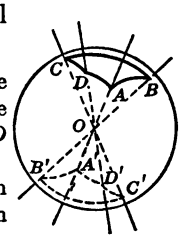
A pair of gloves are analogous to two symmetric polyedral angles. All parts of one are equal to corresponding parts of the other, but are arranged in reverse order. Right and left parts of animals are symmetric to each other in this sense. Numerous such illustrations of symmetry can be found in nature and in the arts.

837. Two spherical polygons on the same sphere or an equal spheres are **symmetric** if their corresponding central polyedral angles are symmetric.



In the figure, spherical triangles ABC and $A'B'C'$ are symmetric for their corresponding central triedral angles are symmetric.

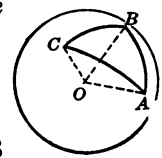
838. The two polyedral angles formed by a pyramidal surface are symmetric; and, if a sphere is drawn with its center at the vertex of the pyramidal surface, then the pyramidal surface will intersect the sphere in two symmetric spherical polygons.



It is evident that the parts of the two nappes of the pyramidal surface with vertex O are arranged in reverse order, then the corresponding spherical polygons $ABCD$ and $A'B'C'D'$ are symmetric.

Of course, symmetric spherical polygons can lie in any position on the sphere, for a figure on a sphere can be moved about without changing its shape.

839. Theorem. *In any spherical triangle, the sum of two sides is greater than the third side.*



Given spherical $\triangle ABC$ on sphere O.

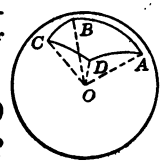
To prove that $\overset{\frown}{AB}+\overset{\frown}{BC}>\overset{\frown}{AC}$.

Proof. $\angle AOB+\angle BOC>\angle AOC$. § 608

But $\angle AOB=^\circ\overset{\frown}{AB}$, $\angle BOC=^\circ\overset{\frown}{BC}$, and $\angle AOC=^\circ AC$. Why?

$\therefore \overset{\frown}{AB}+\overset{\frown}{BC}>\overset{\frown}{AC}$. § 111

840. Theorem. *In any convex spherical polygon, the sum of the sides is less than a great circle of the sphere, or less than 360°.*



$\angle AOB+\angle BOC+\angle COD+\angle DOA<360^\circ$. § 609

$\therefore \overset{\frown}{AB}+\overset{\frown}{BC}+\overset{\frown}{CD}+\overset{\frown}{DA}<360^\circ$. Why?

EXERCISES

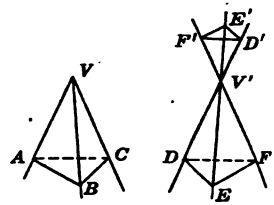
1. Prove that any side of a spherical polygon is less than the sum of the remaining sides.

2. Show that any side of a convex spherical polygon is less than 180°.

3. Given a spherical $\triangle ABC$, with $AB=57^\circ$ and $AC=125^\circ$. Between what limits must BC lie?

4. Given a spherical quadrilateral three of whose sides are 77°, 98°, and 45° respectively. Between what limits must the fourth side lie? Between what limits must the fourth side lie if three sides are 27°, 33°, and 100° respectively?

841. Theorem. *Two triedral angles are congruent or symmetric if a face angle and the two adjacent diedral angles of one are equal respectively to a face angle and the two adjacent diedral angles of the other.*



Given triedral $\measuredangle V\text{-}ABC$ and $V'\text{-}DEF$, with $\angle AVC = \angle DV'F$, and the adjacent diedral angles equal.

To prove that the triedral angles are congruent or symmetric.

Proof. If parts are arranged in the same order, superpose as in the analogous case in plane geometry (§ 248). If parts are not in the same order, extend the edges through the vertex of one and superpose the other upon the symmetric triedral angle thus formed. § 836

842. Theorem. *On the same sphere or on equal spheres, two spherical triangles are congruent or symmetric if a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

Construct corresponding central triedral angles and apply the previous theorem.

843. Theorem. *Two isosceles symmetric spherical triangles are congruent.*

844. Theorem. *Two triedral angles are congruent or symmetric if two face angles and the included diedral angle of one equal respectively to two face angles and the included diedral angle of the other.*

Superpose if arranged in the same order. If arranged in opposite order, superpose one upon the triedral angle symmetric to the other.

845. Theorem. *On the same sphere or on equal spheres, two spherical triangles are congruent or symmetric if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

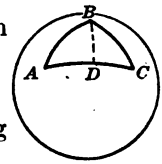
Construct corresponding central triedral angles and apply the previous theorem.

846. Theorem. *On the same sphere or on equal spheres, two spherical triangles are congruent or symmetric if the three sides of one are equal respectively to the three sides of the other.*

Construct the corresponding central triedral angles and use § 610 when parts are in the same order. When parts are in the opposite order, construct a triedral angle symmetric to one by § 836, and then apply § 610.

847. A triedral angle is **isosceles** if it has two equal face angles. It is equilateral if it has all its face angles equal.

848. Theorem. *If a spherical triangle is isosceles, the angles opposite the equal sides are equal.*



Given the isosceles spherical $\triangle ABC$, with $\overset{\frown}{AB} = \overset{\frown}{CB}$.

To prove $\angle A = \angle C$.

Proof. Draw the great circle arc BD bisecting $\overset{\frown}{AC}$.

Then spherical $\triangle ABD$ and CBD are congruent or symmetric.

§ 846

$\therefore \angle A = \angle C.$ Why?

EXERCISES

1. If a triedral angle is isosceles, the diedral angles opposite the equal face angles are equal.

2. If a spherical triangle is equilateral, it is equiangular.

3. If a triedral angle is equilateral, all the diedral angles are equal.

4. State and prove theorems on the sphere corresponding to the following theorems in plane geometry:

(1) All points in the perpendicular bisector of a straight line are equidistant from the extremities of the line. § 205

(2) All points equidistant from the extremities of a straight line lie in the perpendicular bisector of the line. § 206

(3) All points in the bisector of an angle are equidistant from the sides of the angle. § 199

5. In the isosceles triangle of § 848, could $\overset{\frown}{AB}$ and $\overset{\frown}{CB}$ each be greater than 90°? Could each be greater than 180°? Could $\angle ABC$ be 20° and $\overset{\frown}{AC}$ be 5°? Could $\angle ABC$ be 20° and $\overset{\frown}{AC}$ be 20°? Illustrate on a globe.

849. A triedral angle is called a **rectangular,** a **birectangular,** or a **trirectangular** triedral angle, according as it has one, two, or three right diedral angles.

Three concurrent faces of a rectangular solid form a trirectangular triedral angle. Three concurrent faces of a right prism with an oblique triangular base form a birectangular triedral angle.

850. A spherical triangle is called a **right,** a **birectangular,** or a **trirectangular** spherical triangle, according as it has one, two, or three right spherical angles.

851. The following are derived from §§ 779 and 785:

(1) *In a birectangular spherical triangle, the sides opposite the equal angles are quadrants, and conversely.*

In the figure, O is a birectangular triedral angle, and APB is a birectangular spherical triangle. The faces, or arcs, that pass through P form right angles with the other face, or arc.

(2) *In a trirectangular spherical triangle, the sides are quadrants, and conversely.*

In the figure, O is a trirectangular triedral angle, and APB is a trirectangular spherical triangle. Here each vertex is the pole of the opposite side of the spherical triangle. § 785

852. Theorem. *Three mutually perpendicular planes passed through the center of a sphere divide the sphere into eight congruent trirectangular spherical triangles.*

EXERCISES

1. Find the area of a trirectangular spherical triangle on a sphere having a radius of 10 in.

2. Find the area of a birectangular spherical triangle with a vertex angle of 60°, on a sphere 40 ft. in diameter.

3. What are the volumes cut out of the sphere by the corresponding central triedral angles in Exercises 1 and 2?

4. On a sphere having a radius of 12 in. two sides of an isosceles spherical triangle are quadrants. Find the length of the third side if the vertex angle is 60°.

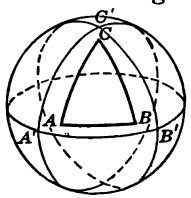
5. On a sphere having a radius of 4 in. the sides of a spherical triangle are 80°, 140°, and 120° respectively. Find the length of each side.

6. Find the area of a birectangular spherical triangle having one angle 1°, on a sphere of radius r.

7. What is the volume cut out of the sphere by the central triedral angle corresponding to the spherical triangle described in Exercise 6?

POLAR TRIANGLES

853. If from the vertices of the spherical triangle ABC as poles, great circles are drawn, the sphere is divided into eight spherical triangles. If the intersection of the circles having A and B as poles, that is on the same side of AB as C, is labeled C', and similarly for the vertices A' and B'; then the spherical triangle $A'B'C'$ is the **polar triangle** of ABC.



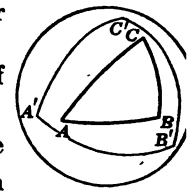
"On the same side of" means that C and C' are both in one of the hemispheres formed by the great circle of which AB is an arc.

Two spherical triangles related as ABC and $A'B'C'$ are spoken of as **polar triangles.**

Other spherical triangles should be constructed, some having sides of few degrees and others having sides near 180°. The polar triangles of these should be drawn and labeled as directed.

It is important that these relations should be clearly understood, as they are used in proving many theorems.

854. Theorem. ***If one spherical triangle is the polar of another, then the second is the polar of the first.***



Given the spherical $\triangle ABC$ and its polar $\triangle A'B'C'$.

To prove that $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

Proof. A is the pole of $\overset{\frown}{B'C'}$, and C is the pole of $\overset{\frown}{A'B'}$. Given

Then B' is at a quadrant's distance from both A and C. Why

Hence B' is the pole of AC. § 7

Similarly A' is the pole of $\overset{\frown}{BC}$, and C' the pole of $\overset{\frown}{AB}$.

$\therefore$ $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$. § 8

EXERCISES

1. One side of a spherical triangle on the earth's surface is on t equator. Prove that a vertex of the polar triangle is either at the nor pole or at the south pole of the earth.

2. Prove the theorem of § 854 by taking as given that $\triangle ABC$ is t polar triangle of $\triangle A'B'C'$.

3. If a vertex of a spherical triangle is at the pole of the great circle which one side is an arc, prove that one vertex of the polar triangle is this pole and that one side is on this great circle.

4. Draw the following spherical triangles and their polar triang

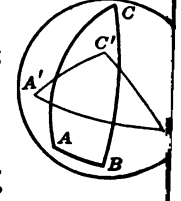
(1) A triangle each side of which is less than a quadrant.

(2) A triangle each side of which is greater than a quadrant.

(3) A triangle only one side of which is greater than a quadrant.

(4) A birectangular triangle.

(5) A trirectangular triangle.



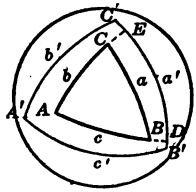
NOTE. A slated sphere should be used in making these constructions. A pair of compasses should be so adjusted that they will always strike an arc of a great circle on the sphere being used.

5. Can the sides of a spherical triangle intersect the sides of its polar triangle? Under what conditions?

6. Prove the theorem of § 854 by using this figure.

7. Prove that a trirectangular triangle is its own polar triangle.

855. Theorem. ***In two polar triangles, each angle of the one is equal in degrees to the supplement of the side opposite to it in the other.***



Given the polar triangles ABC and $A'B'C'$, with the letter at each vertex of an angle denoting its value in degrees, and the letters a, b, c and a', b', c' denoting the values of the opposite sides in degrees.

To prove that $A+a'=180°$, $B+b'=180°$, $C+c'=180°$;

and $A'+a=180°$, $B'+b=180°$, $C'+c=180°$.

Proof. Let D and E respectively be the intersections of $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$, produced if necessary, with $\overset{\frown}{B'C'}$.

Then	$\overset{\frown}{C'D}=90°$, and $\overset{\frown}{EB'}=90°$.	§ 785
Or	$\overset{\frown}{C'E}+\overset{\frown}{ED}=90°$, and $\overset{\frown}{ED}+\overset{\frown}{DB'}=90°$.	Why?
Then	$\overset{\frown}{C'E}+\overset{\frown}{ED}+\overset{\frown}{ED}+\overset{\frown}{DB'}=180°$.	Why?
But	$\overset{\frown}{C'E}+\overset{\frown}{ED}+\overset{\frown}{DB'}=a'$, and $\overset{\frown}{ED}=°\angle A$.	§§ 109, 829
	$\therefore A+a'=180°$.	Why?

Similarly $B+b'=180°$, and $C+c'=180°$.

Complete the proof by starting with $\triangle A'B'C'$.

EXERCISES

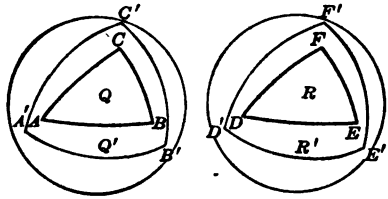
1. If the sides of a spherical triangle have respectively 70°, 80°, and 110°, find the values of the angles of its polar triangle.

2. Could the sides of a spherical triangle be each 8°? What would then be the value of the angles in the polar triangle?

3. Could the angles of a spherical triangle be 8°, 9°, and 10° respectively? What would then be the sides of the polar triangle?

4. Find the area of the triangle that is the polar of a birectangular spherical triangle having an angle of 1°, on a sphere 4 ft. in radius.

856. Theorem. *If two spherical triangles on the same sphere or on equal spheres are mutually equiangular, they are mutually equilateral, and are either congruent or symmetric.*



Given two mutually equiangular spherical triangles Q and R on equal spheres.

To prove that Q and R are mutually equilateral, and are either congruent or symmetric.

Proof. Construct $\triangle Q'$ and R', the polar triangles of $\triangle Q$ and R respectively.

	$\triangle Q$ and R are mutually equiangular.	Given
Then	$\triangle Q'$ and R' are mutually equilateral.	§ 855
Hence	$\triangle Q'$ and R' are either congruent or symmetric.	§ 846
And	$\triangle Q'$ and R' are mutually equiangular.	Why?
But	$\triangle Q$ and R are the polar triangles of $\triangle Q'$ and R'.	§ 854
Then	$\triangle Q$ and R are mutually equilateral.	§ 855
	$\therefore$ $\triangle Q$ and R are either congruent or symmetric.	§ 846

857. Theorem. *If two triedral angles have their diedral angles respectively equal, their face angles are respectively equal, and they are either congruent or symmetric.*

EXERCISES

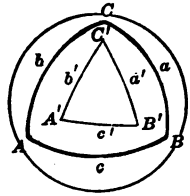
1. Is there a theorem in plane geometry analogous to the theorems of §§ 856, 857?

2. If two angles of a spherical triangle are equal, the triangle is isosceles.

3. If a spherical triangle is equiangular it is equilateral.

4. If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the greater side is opposite the greater angle; and conversely. Proof of theorem is similar to that of § 184.

858. Theorem. ***The sum of the angles of a spherical triangle is greater than 180° and less than 540°.***



Given the spherical $\triangle ABC$ with the letter at each vertex of an angle denoting its value in degrees, and the letters a, b, c denoting the values of the opposite sides in degrees.

To prove $A+B+C>180°$ and $<540°$.

Proof. Construct the polar $\triangle A'B'C'$ of $\triangle ABC$.

Then	$A+a'=180°$, $B+b'=180°$, $C+c'=180°$.	§ 855
Hence	$A+B+C+a'+b'+c'=540°$.	§ 105
But	$a'+b'+c'<360°$.	§ 840
	$\therefore A+B+C>180°$.	§ 178
Also	$a'+b'+c'>0°$.	Why?
	$\therefore A+B+C<540°$.	Why?

859. Theorem. *The sum of the diedral angles of a triedral angle is greater than 180° and less than 540°.*

EXERCISES

1. How would you draw a triangle on a sphere, the sum of the angles of which would be only a little greater than 180°? Only a little less than 540°? Could you draw such triangles on a small sphere or would it be necessary to have a large sphere?

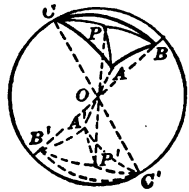
2. What is the locus of points on a sphere and equally distant from two intersecting arcs of great circles of the sphere?

3. What is the locus of the points on a sphere that are equally distant from the ends of an arc of a great circle?

4. If the sides of a spherical triangle each have less than 3°, what can be said of the angles of its polar triangle? Could the angles of a spherical triangle be respectively 170°, 6°, and 12°?

AREAS OF SPHERICAL POLYGONS

860. Theorem. *On the same sphere or on equal spheres two symmetric spherical triangles are equal in area.*



Given the two symmetric spherical triangles ABC and $A'B'C'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. Place the two triangles so that they are formed by the same pyramidal surface with its vertex at O. § 838

Let P be the pole of a small circle passing through A, B, and C.

Draw the diameter POP' and the great circle arcs PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$.

Then $\overset{\frown}{PA} = \overset{\frown}{PB} = \overset{\frown}{PC}$. § 783

But $\overset{\frown}{P'A'} = \overset{\frown}{PA}$, $\overset{\frown}{P'B'} = \overset{\frown}{PB}$, $\overset{\frown}{P'C'} = \overset{\frown}{PC}$. Why?

Then $\overset{\frown}{P'A'} = \overset{\frown}{P'B'} = \overset{\frown}{P'C'}$. Why?

And $\triangle PAB$ and $P'A'B'$ are isosceles as well as symmetric.

Hence $\triangle PAB \cong \triangle P'A'B'$. § 843

Similarly $\triangle PBC \cong \triangle P'B'C'$, and $\triangle PCA \cong \triangle P'C'A'$.

Then $\triangle PAB + \triangle PBC + \triangle PCA = \triangle P'A'B' + \triangle P'B'C' + \triangle P'C'A'$. § 105

$\therefore \triangle ABC = \triangle A'B'C'$. §§ 109, 104

EXERCISES

1. Draw a figure in which the pole P, § 860, falls outside the triangle ABC. What difference would this make in the proof?

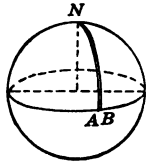
2. A sphere can be divided into four, eight, or twenty equal equilateral spherical triangles.

861. Spherical Excess. The excess of the sum of the angles of a spherical triangle over 180° is called the **spherical excess of the triangle.**

The **spherical excess of a spherical polygon** of n sides is the excess of the sum of its angles over $(n-2)180°$, that is, it is the excess over the sum of the angles of a plane polygon of the same number of sides.

862. Area of a Spherical Triangle. The sides as well as the angles of a spherical triangle are usually given in degrees. It is evident then that the number of square units in the area of a spherical triangle will depend upon the radius of the sphere on which the triangle is. When the radius of the sphere is not given it is convenient to have a unit of measure that is some fractional part of the surface of a sphere.

863. Spherical Degree. One half the area of a lune of 1°, or the area of a birectangular triangle having a vertex angle of 1°, or $\frac{1}{720}$ of the surface of a sphere, is called a **spherical degree.** It is the unit of measure of spherical polygons.



A polyedral angle whose vertex is at the center of a sphere cuts a spherical polygon out of the sphere. The area of this polygon in spherical degrees is called the measure of the polyedral angle. In this manner the magnitudes of polyedral angles may be compared.

EXERCISES

1. How many spherical degrees in a lune of 20°? Of 46° 30′?

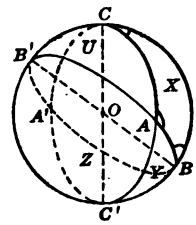
2. How many spherical degrees in a trirectangular spherical triangle? What is the area in square inches of a trirectangular triangle on a sphere having a radius of 10 in.?

3. How many square feet in the area of a birectangular triangle with vertex angle 18° on a sphere 6 ft. in. diameter?

4. Find the spherical excess in spherical triangles having the following angles: (1) 100°, 117°, 136°. (2) 141°, 97° 30′, 67° 45′. (3) 116° 23′ 14″, 127° 43′ 18″, 69° 45′ 15″.

5. Find the spherical excess in a spherical pentagon having angles of 113°, 97°, 119° 17′, 141° 23′, and 172° 27′.

864. Theorem. *A spherical triangle is equivalent to a lune on the same sphere, whose angle is half the spherical excess of the triangle.*



Given the spherical $\triangle ABC$ on a sphere of surface S and center O, with A, B, and C denoting the value in degrees of the angles at the respective vertices.

To prove that $\triangle ABC$ = a lune of angle $\frac{1}{2}(A+B+C-180°)$.

Proof. Complete the great circles of the sides of $\triangle ABC$, and let one of them as BC from the boundary of the hemisphere which is divided into four triangles, ABC, or X, ABC' or Y, $AB'C'$ or Z, and $AB'C$ or U.

	$\triangle BA'C$ on the reverse hemisphere $= \triangle Z$.	§ 860
	Lune of $\angle A = \triangle X + \triangle Z$.	Why?
	Lune of $\angle B = \triangle X + \triangle U$.	Why?
	Lune of $\angle C = \triangle X + \triangle Y$.	Why?
Then	lune of $\angle(A+B+C) = 3\triangle X + \triangle Z + \triangle U + \triangle Y$.	§ 105
But	$\triangle X + \triangle Z + \triangle U + \triangle Y = \frac{1}{2}S$ = lune of $\angle 180°$.	Why?
Then	lune of $\angle(A+B+C) = 2\triangle X$ + lune of $\angle 180°$.	§ 111
Or	$2\triangle X$ = lune of $\angle(A+B+C-180°)$.	Why?
	$\therefore \triangle ABC$ = lune of $\angle\frac{1}{2}(A+B+C-180°)$.	Why?

865. Theorem. *On the same sphere or on equal spheres, the areas of two spherical triangles are to each other as their spherical excesses.*

866. Theorem. *If A denotes the area of a spherical triangle, E the number of degrees in its spherical excess, and r the radius of the sphere, then the area is given by the formula:* $A = \dfrac{\pi r^2 E}{180}$.

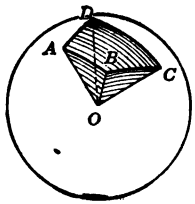
867. Theorem. *A spherical polygon is equivalent to a lune on the same sphere, whose angle is half the spherical excess of the polygon.*

Divide the spherical polygon into spherical triangles by drawing the diagonals from one vertex. Then the area of the polygon is equivalent to the sum of the areas of the triangles, and the spherical excess of the polygon is equal to the sum of the excesses of the triangles.

868. Theorem. *The area of a spherical polygon is given by the formula* $A = \frac{\pi r^2 E}{180}$, *where* r *is the radius of the sphere and* E *the number of degrees in the spherical excess.*

NOTE. This theorem was stated by A. Girard in 1629 and was first completely proved by Cavalieri a few years later.

869. Spherical Pyramid. The portion of the volume of a sphere bounded by a spherical polygon and the planes of its sides is called a **spherical pyramid.**



The polygon is the **base** of the spherical pyramid, and the center of the sphere is its **vertex.**

Thus, *O-ABCD* is a spherical pyramid.

870. Theorem. *The volume of a spherical pyramid is equal to one-third the product of the area of its base by the radius of the sphere.*

EXERCISES

1. Find the areas of triangles on spheres of the given radii, and having angles as follows:

(1) 80°, 140°, 120°, $r = 10$ in. (3) 125°, 135°, 145°, $r = 50$ ft.

(2) 150°, 75°, 110°, $r = 16$ in. (4) 179°, 179°, 179°, $r = 10$ ft.

(5) 112° 30′ 15″, 127° 43′ 30″, 175° 27′ 15″, $r = 20$ ft.

(6) 37° 43′ 27″, 15° 45′ 34″, 170° 45′ 43″, $r = 4000$ mi.

2. Find the areas of polygons on spheres of the given radii, and having angles as follows:

(1) 70°, 170°, 100°, 125°, $r = 12$ in.

(2) 100°, 98°, 175°, 160°, 128°, 96°, $r = 8$ ft.

(3) 75° 45′ 17″, 127° 14′ 16″, 102° 43′ 16″, 150° 43′ 41″, $r = 20$ in.

3. Find the area of the triangle on the earth's surface, determined by the meridians at 48° and 75° west longitude, and the equator. Use 4000 miles for the radius of the earth.

4. Find the volume of a spherical pyramid, given that its base is an equiangular triangle having each angle 120°, on a sphere of radius 9 in.

5. Find the volume of a spherical pyramid, given that its base is a spherical hexagon each angle of which is 130°, on a sphere of 12 in. radius.

6. The volume of a spherical pyramid whose base is an equiangular spherical triangle with angles of 105° is 256π cu. in. Find the diameter of the sphere.

7. What is the greatest area that a triangle on the surface of the earth may have if the sum of its angles is not to differ by more than 1° from 180°? If the sum is not to differ by more than 1′ from 180°?

8. A triangle having an area of 62 sq. in. is on a sphere that has an area of 500 sq. in. Two of its angles are 92° and 135° respectively. Find the third angle.

9. The sides of a spherical triangle are 79°, 88°, and 115°, and it is on a sphere whose radius is 18 in. Find the area of its polar triangle.

QUESTIONS

1. Define a circle. A sphere. Is a circle an area? Is a sphere a solid? Define a sphere as a locus. Define a circle as a locus.

2. Is it in agreement with the definition of a sphere to speak of the area of a sphere? What is the volume of a sphere? Compare the definition of the area of a circle with that of the volume of a sphere.

3. What is a spherical angle? A spherical triangle? Is a triangle an area?

4. How is a plane angle measured? A spherical angle? A diedral angle? A polyedral angle?

5. What is a lune? A wedge? How may lunes be compared in size? How may wedges be compared in size?

6. Define the sector of a sphere, and compare with the sector of a circle.

7. Define the segment of a sphere, and compare with the segment of a circle. Do we speak of the segment of a circle of one base? Is there any reason for not defining the segment of a circle so that we can speak of a segment of two bases?

8. Is the distance between two points on a sphere defined so that it is analogous to the distance between two points in a plane?

9. Explain why it is well to treat spherical polygons in connection with polyedral angles. State the relations between the parts of a spherical triangle and the parts of the corresponding polyedral angle.

10. State several theorems about spherical triangles that are analogous to theorems about plane triangles. Is a plane triangle determined when its angles are known? Is a spherical triangle determined when its angles are known?

11. What is a polar triangle? What are some relations between a spherical triangle and its polar triangle? State a striking difference between the angles of a spherical triangle and those of a plane triangle.

12. How many points determine a circle? A sphere?

13. How many points on a sphere determine a small circle? A great circle?

14. Define axis, pole, polar distance.

15. If a great circle is perpendicular to a small circle, what must it contain? Is the converse of this true?

16. Give an outline of the steps in determining the formulas for the area of a sphere. State these formulas, and state when each is the best one to use.

17. Do the same for the volume of a sphere.

18. What is the formula for the area of a zone? A lune? A spherical triangle? Can you give a practical application of the use of each of these areas?

19. What is the formula for the volume of a spherical segment? A wedge? A spherical pyramid? A spherical sector? Can you give a practical application of the use of each of these volumes?

20. What remarkable relation is there between the volumes of a cylinder, sphere, and cone? Is there a similar relation between their areas?

GENERAL EXERCISES

1. The volume of any polyedron circumscribed about a sphere is equal to the area of its surface times one-third the radius of the sphere.

2. The volumes of polyedrons circumscribed about the same sphere are to each other as their surfaces.

3. What is the locus of all points in space equally distant from two given points and at a distance r from a third given point?

4. Show how to find the pole of a given circle on a sphere.

5. Construct a circle through three given points on a sphere.

6. Four balls of equal radii r are placed on a plane and touching each other in such a way that their centers form a square. A fifth ball of the same radius is placed upon the four first balls. Find the distance of its center from the plane.

7. A cylinder is inscribed in a sphere of radius r and has a lateral area equal to one-half the area of the sphere. Find the radius of the cylinder. *Ans.* $\frac{1}{2}r\sqrt{2}$.

8. A zone of one base is a mean proportional between the remaining part of the area of the sphere and the total area of the sphere. How far is its base from the center of the sphere?

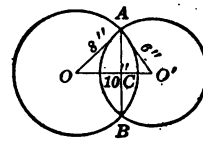
9. A spherical segment of one base is half as large as the spherical sector to which it belongs. If the radius of the sphere is 4 in., find the altitude of the segment.

10. The inside of a glass is in the form of a right circular cone whose vertex angle is 60° and the diameter of whose base is 4 in. The glass is filled with water and the largest sphere that can be immersed in the water is placed in the glass. Find the volume of water remaining in the glass.

11. An irregular portion, but less than half, of a material sphere, is given. Show how to construct the diameter of the sphere.

12. In the figure of § 813, the plane MNP is tangent to the sphere inscribed in the cube. If the edge of the cube is a find DN.

13. Two spheres whose radii are respectively 6 in. and 8 in. have their centers 10 in. apart. Find the volume of the portion common to the two spheres. This is the form of a spherical lens.



14. A hole of diameter 4 in. was bored through the center of a sphere of diameter 12 in. Find the volume of the part cut away.

15. A sphere is inscribed in a right circular cylinder having an altitude equal to the diameter of the sphere. Two planes parallel to the base of the cylinder cut the cylinder and the sphere. Prove that the surface of the cylinder lying between the planes equals the area of the zone between the planes.

16. A circle is circumscribed about an equilateral triangle. Find the ratios of the areas and the volumes of the solids formed by revolving the triangle and the circle about an altitude of the triangle as axis.

17. A hollow copper sphere used as a float in water weighs 10 oz., and has a diameter of 5 in. How heavy a weight will it support?

Ans. Less than 27.9 oz.

FORMULAS FOR REFERENCE

Lateral area of prism. $S=pe$, $S=ph$.

Lateral area of circular cylinder. $S=pe$, $S=ch$.

Volume of rectangular parallelepiped. $V=abh$.

Volume of parallelepiped. $V=Bh$.

Volume of prism or cylinder. $V=Bh$.

Volume of hollow circular cylinder. $V=\frac{1}{4}\pi h(D+d)(D-d)$.

Lateral area of regular pyramid. $S=\frac{1}{2}ps$.

Lateral area of right circular cone. $S=\frac{1}{2}cs=\pi rs$.

Total area of right circular cone. $T=\pi rs+\pi r^2=\pi r(s+r)$.

Lateral area of frustum of regular pyramid. $S=\frac{1}{2}s(P+p)$.

Lateral area of frustum of right circular cone. $S=\pi(R+r)s$.

Volume of pyramid or cone. $V=\frac{1}{3}Bh$.

Volume of circular cone. $V=\frac{1}{3}\pi r^2h$.

Volume of frustum of pyramid or cone.

$$V=\tfrac{1}{3}h(B_1+B_2+\sqrt{B_1B_2}).$$

Volume of frustum of circular cone.

$$V=\tfrac{1}{3}\pi h(r_1^2+r_2^2+r_1r_2)=\tfrac{1}{12}\pi h(d_1^2+d_2^2+d_1d_2).$$

Volume of prismatoid. $V=\frac{1}{6}h(B_1+B_2+4M)$.

Area of a sphere. $S=4\pi r^2$.

Area of a zone. $Z=2\pi rh$.

Area of a lune. $L=\dfrac{u}{360}(4\pi r^2)$.

Volume of a sphere. $V=\frac{4}{3}\pi r^3=\frac{1}{6}\pi d^3=\frac{1}{3}rS$.

Volume of spherical segment. $V=\frac{1}{2}\pi h(r_1^2+r_2^2)+\frac{1}{6}\pi h^3$.

Volume of spherical wedge. $V=\dfrac{u}{360}(\frac{4}{3}\pi r^3)$.

Area of spherical polygon. $A=\dfrac{\pi r^2E}{180}$.

USEFUL NUMBERS

1 cu. ft. of water weighs 62.5 lb. (approx.) = 1000 oz.

1 gal. of water weighs $8\frac{1}{3}$ lb. (approx.).

1 atmosphere pressure = 14.7 lb. per sq. in. = 2116. lb. per sq. ft.

1 atmosphere pressure = 760 mm. of mercury.

A column of water 2.3 ft. high = a pressure of 1 lb. per sq. in.

1 Kg. = 2.2 lb. (approx.).

1 gal. = 231 cu. in. (by law of Congress).

1 cu. ft. = $7\frac{1}{2}$ gal. (approx.) or better 7.48 gal.

1 cu. ft. = $\frac{4}{5}$ bu. (approx.).

1 bbl. = 4.211− cu. ft. (approx.).

1 bu. = 2150.42 cu. in. (by law of Congress) = 1.24446− cu. ft.

1 bu. = $\frac{5}{4}$ cu. ft. (approx.).

1 perch = $24\frac{3}{4}$ cu. ft. but usually taken 25 cu. ft.

1 in. = 25.4 mm. (approx.).

1 m. = 39.37 in. (by law of Congress).

1 lb. (avoirdupois) = 7000 grains (by law of Congress).

1 lb. (troy or apothecaries) = 5760 grains.

$\pi = 3.14159265358979+ = 3.1416 = \frac{3}{1}\frac{5}{1}\frac{5}{3} = 3\frac{1}{7}$ (all approx.).

$\sqrt{2} = 1.4142136.$ $\sqrt{3} = 1.7320508.$

$\sqrt{5} = 2.2360680.$ $\sqrt{6} = 2.4494897.$

$\sqrt[3]{2} = 1.2599210.$ $\sqrt[3]{3} = 1.4422496.$

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